

On the monotone stability approach to BSDEs with jumps: Extensions, concrete criteria and examples*

Dirk Becherer[†]Martin Büttner[†]Klebert Kentia[‡]

Abstract

We extend the monotone stability approach for backward stochastic differential equations (BSDEs) that are jointly driven by a Brownian motion and a random measure, which can be of infinite activity and time-inhomogeneous with non-deterministic compensator. The BSDE generator function can be non-convex and needs not to satisfy classical global Lipschitz conditions in the jump integrand. We contribute concrete criteria, that are easy to verify, and extended results for comparison and for existence and uniqueness of bounded solutions to BSDEs with jumps. The scope of results, applicability of assumptions and differences to related results by some alternative approaches are demonstrated by several examples for control problems from finance.

Keywords: Backward stochastic differential equations, random measures, monotone stability, Lévy processes, utility maximization, good deal bounds

MSC2010: 60G57, 60H20, 93E20, 60G51, 91G80

1 Introduction

We study bounded solutions (Y, Z, U) to backward stochastic differential equations with jumps

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de),$$

which are jointly driven by a Brownian motion B and a compensated random measure $\tilde{\mu} = \mu - \nu^{\mathbb{P}}$ of some integer-valued random measure μ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Classical BSDE theory [PP90] studies square integrable solutions to BSDEs driven solely by Brownian motion B under global Lipschitz assumptions on the generator f . An important extension considers generators with quadratic growth in Z , for which [Kob00] derived bounded solutions by pioneering a monotone stability approach, and [Tev08] by a fixed point approach. Another extension are BSDEs with jumps (JBSDEs) on non-Brownian filtrations, which are driven additionally by a compensated random measure $\tilde{\mu}$. Such JBSDEs involve a further stochastic integral w.r.t. $\tilde{\mu}$ whose integrand U , differently from Z , takes values in a typically infinite dimensional function space instead of an Euclidean space. Square integrable solutions under global Lipschitz conditions for JBSDEs were obtained in [TL94, BBP97] in a setting of Poisson random measures. For square integrable solutions to JBSDEs with generators having a monotonicity in the Y -component, see [Par97, Roy06]; Bounded solutions to JBSDEs were obtained in [Bec06] for a (possibly non-homogeneous) random measure for jumps of finite activity and for generators f that do not need to be uniformly Lipschitz in U . In the context of non-Lipschitz quadratic generators (also in Z), JBSDEs have been studied by [Mor09, Mor10] by a monotone stability approach for a specific generator (related to exponential

*Support from German Science Foundation DFG via Berlin Mathematical School, RTG 1845 StoA, and research center MATHEON is gratefully acknowledged. A previous version of this paper was circulated under the title 'bounded solutions for BSDEs with jumps of infinite activity'.

[†]Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany.
Emails: becherer, buettner@math.hu-berlin.de

[‡]Previous: Institut für Mathematik, Humboldt-Universität zu Berlin, Germany.

Current: Institut für Mathematik, Goethe-Universität Frankfurt, D-60054 Frankfurt am Main, Germany.
Email: kentia@math.uni-frankfurt.de

utility), by [EMN16] using a quadratic-exponential semimartingale approach from [BEK13], and by [LS14] or [KTPZ15] with again different approaches, relying on duality methods or, respectively, the fixed-point idea of [Tev08] for quadratic BSDEs.

It is known [BBP97, Roy06, CE10] that comparison theorems for BSDEs with jumps require more delicate technical conditions than in the Brownian case. We first slightly generalize the seminal but technical (\mathbf{A}_γ) -condition for comparison of JBSDEs from [Roy06]. Our first contribution is the extension of comparison, existence and uniqueness results for bounded solution of JBSDE to the infinite activity case for a family (2.6) of generators, that do not need to be Lipschitz in the U -argument (but are in Z). This shows how the scope of the monotone stability approach to JBSDE, pioneered by [Mor09, Mor10] for one particular generator, can be advanced and how the strong approximation step can be argued in concisely a setting, which may be of particular appeal in a pure jump case without a Brownian motion, cf. Corollary 4.12. To be useful for further research and towards applications, our second contribution is to derive sufficient concrete criteria for comparison and wellposedness that are comparably easy to verify in actual applications, as they are formulated in terms of properties of certain functions for generators f from the family (2.6) w.r.t. basically Euclidean arguments. This is the main trust for our comparison theorems in Section 3 and the wellposedness Theorem 4.13, (compared to e.g. Theorem 4.11 whose conditions are more general and more abstract). A third contribution are several application examples which illustrate the scope and applicability of our results, and pinpoint some differences to interesting alternative studies on JBSDEs, like [LS14, KTPZ15], which are complementary in using other methods and different kinds of assumptions (see e.g. Section 5.1.2 and Remarks 4.15-4.16).

Our approach can be described precisely as follows: The comparison theorem gives rise to a-priori estimates on the L^∞ -norm for the Y -component of the JBSDE solution. This step enables quick intermediate wellposedness results for JBSDE with finite jump activity. To advance from here to infinite activity, we approximate the generator f by a monotone sequence of generators for which solutions do exist, extending the monotone stability approach from [Kob00] and [Mor09, Mor10] (for a particular JBSDE). At first, such an argument only works for terminal conditions ξ being small enough in the L^∞ -norm. But by pasting solutions for sufficiently small terminal conditions one can thereafter show convergence to the bounded solution of the BSDE for the original data (ξ, f) .

For the present paper, the compensator $\nu(\omega, dt, de)$ of $\mu(\omega, dt, de)$ can be stochastic and does not need to be a product measure like $\lambda(de) \otimes dt$, as it would be natural e.g. in a Lévy-process setting, but it is allowed to be inhomogeneous in that it can vary predictably with (ω, t) . In this sense, ν is only assumed to be absolutely continuous to some reference product measure $\lambda \otimes dt$ with λ being σ -finite, see equation (2.1). Such appears useful, but requires extra effort in the specification of generator properties, cf. Section 2. An essential property on the filtration is that $\tilde{\mu}$ jointly with B (or alone) satisfies the property of weak predictable representation for martingales, see (2.2). As explained in Example 2.1, our setup permits for stochastic dependencies between B and $\tilde{\mu}$, which can be relevant for modeling of applications (cf. examples in [BS05]). This encompasses many interesting driving noises for BSDEs, including Lévy processes, Poisson random measures, marked point processes, (semi-)Markov chains or much more general step processes, connecting to a wide range of literature, e.g. [NS01, CE10, CFJ16, GL16, GS16]. For results on JBSDE in interesting other directions, let us mention [CM08] for comparison of JBSDEs with (doubly) reflection for Lipschitz generators, [KMPZ10] for (minimal) solutions with constraints on jumps for Poisson random measures of finite activity, and [DI10] for time delayed generators.

The paper is organized as follows. Section 2 introduces the setting and mathematical background. In Sections 3 and 4, we prove comparison results and show existence as well as uniqueness for bounded solutions to JBSDEs, both for finite and infinite activity of jumps. Last but not least, Section 5 illustrates the scope and applicability of our results by solving several control problems from finance, explaining also some differences to related recent literature in concrete examples.

2 Preliminaries

This section presents the technical framework, sets notations and discusses key conditions. First we recall essential facts on stochastic integration w.r.t. random measures and on bounded solutions for Backward SDEs which are driven jointly by Brownian motions and a compensated random measure.

For notions from stochastic analysis not explained here we refer to [JS03, HWY92].

Inequalities between measurable functions are understood almost everywhere w.r.t. an appropriate reference measure, typically \mathbb{P} or $\mathbb{P} \otimes dt$. Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right continuity and completeness, assuming $\mathcal{F}_T = \mathcal{F}$ and \mathcal{F}_0 being trivial (under \mathbb{P}); Thus we can and do take all semimartingales to have right continuous paths with left limits, so-called càdlàg paths. Expectations (under \mathbb{P}) are denoted by $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$. We will denote by \mathbf{A}^T the transpose of a matrix \mathbf{A} and simply write $\mathbf{x}\mathbf{y} := \mathbf{x}^T \mathbf{y}$ for the scalar product for two vectors \mathbf{x}, \mathbf{y} of same dimensionality. Let H be a separable Hilbert space and denote by $\mathcal{B}(E)$ the Borel σ -field of $E := H \setminus \{0\}$, e.g. $H = \mathbb{R}^l$, $l \in \mathbb{N}$ or $H = \ell^2 \subset \mathbb{R}^{\mathbb{N}}$. Then $(E, \mathcal{B}(E))$ is a standard Borel space. In addition, let B be a d -dimensional Brownian motion. Stochastic integrals of a vector valued predictable process Z w.r.t. a semimartingale X , e.g. $X = B$, of the same dimensionality are scalar valued semimartingales starting at zero and denoted by $\int_{(0,t]} Z dX = \int_0^t Z dX = Z \bullet X_t$ for $t \in [0, T]$. The *predictable* σ -field on $\Omega \times [0, T]$ (w.r.t. $(\mathcal{F}_t)_{0 \leq t \leq T}$) is denoted by \mathcal{P} and $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(E)$ is the respective σ -field on $\tilde{\Omega} := \Omega \times [0, T] \times E$.

Let μ be an integer-valued random measure with compensator $\nu = \nu^{\mathbb{P}}$ (under \mathbb{P}) which is taken to be absolutely continuous to $\lambda \otimes dt$ for a σ -finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$ with some $\tilde{\mathcal{P}}$ -measurable, bounded and non-negative density ζ , such that

$$\nu(dt, de) = \zeta(t, e) \lambda(de) dt = \zeta_t d\lambda dt, \quad (2.1)$$

with $0 \leq \zeta(t, e) \leq c_{\nu}$ $\mathbb{P} \otimes \lambda \otimes dt$ -a.e. for some constant $c_{\nu} > 0$. Note that $L^2(\lambda)$ and $L^2(\zeta_t d\lambda)$ are separable Hilbert spaces since λ (and $\lambda_t := \zeta_t d\lambda$) is σ -finite and $\mathcal{B}(E)$ is finitely generated. Since the density ζ can vary with (ω, t) , the compensator ν can be time-inhomogeneous and stochastic. Such permits for a richer dependence structure for $(B, \tilde{\mu})$; For instance, the intensity and distribution of jump heights could vary according to some diffusion process. Yet, it also brings a few technical complications, e.g. function-valued integrand processes U from $\mathcal{L}^2(\tilde{\mu})$ (as defined below) for the JBSDE need not take values in one given L^2 -space (for a.e. (ω, t)), like e.g. $L^2(\lambda)$ if $\zeta \equiv 1$, and the specifications of the domain and of the measurability for the generator functions should take account of such.

For stochastic integration w.r.t. $\tilde{\mu}$ and B we define sets of \mathbb{R} -valued processes

$$\begin{aligned} \mathcal{S}^p &:= \mathcal{S}^p(\mathbb{P}) := \left\{ Y \text{ càdlàg} : |Y|_p := \left\| \sup_{0 \leq t \leq T} |Y_t| \right\|_{L^p(\mathbb{P})} < \infty \right\} \quad \text{for } p \in [1, \infty], \\ \mathcal{L}^2(\tilde{\mu}) &:= \left\{ U \text{ } \tilde{\mathcal{P}}\text{-measurable} : \|U\|_{\mathcal{L}^2(\tilde{\mu})}^2 := \mathbb{E} \left(\int_0^T \int_E |U_s(e)|^2 \nu(ds, de) \right) < \infty \right\}, \end{aligned}$$

and the set of \mathbb{R}^d -valued processes

$$\mathcal{L}^2(B) := \left\{ \theta \text{ } \mathcal{P}\text{-measurable} : \|\theta\|_{\mathcal{L}^2(B)}^2 := \mathbb{E} \left(\int_0^T \|\theta_s\|^2 ds \right) < \infty \right\},$$

where $\tilde{\mu} = \tilde{\mu}^{\mathbb{P}} = \mu - \nu$ denotes the compensated measure of μ (under \mathbb{P}). Recall that for any predictable function U , $\mathbb{E}(|U| * \mu_T) = \mathbb{E}(|U| * \nu_T)$ by the definition of a compensator. If $(|U|^2 * \mu)^{1/2}$ is locally integrable, then U is integrable w.r.t. $\tilde{\mu}$, and $U * \tilde{\mu}$ is defined as the purely discontinuous local martingale with jump process $(\int_E U_t(e) \mu(\{t\}, de))_t$ by [JS03, Def.II.1.27] noting that ν is absolutely continuous to $\lambda \otimes dt$. For $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\tilde{\mu})$ we recall that $Z \bullet B$ and $U * \tilde{\mu} = (U * \tilde{\mu}_t)_{0 \leq t \leq T}$ with $U * \tilde{\mu}_t = \int_0^t \int_E U_s(e) \tilde{\mu}(ds, de)$ are square integrable martingales by [JS03, Thm.II.1.33]. For $Z, Z' \in \mathcal{L}^2(B)$ and $U, U' \in \mathcal{L}^2(\tilde{\mu})$ we have for the predictable quadratic covariations that $\langle U * \tilde{\mu}, U' * \tilde{\mu} \rangle_t = \int_0^t \int_E U_s(e) U'_s(e) \nu(ds, de)$ by [JS03, Thm.II.1.33], $\langle \int Z dB, \int Z' dB \rangle_t = \int_0^t Z_s^T Z'_s ds$ and $\langle \int Z dB, U * \tilde{\mu} \rangle_t = 0$ by [JS03, Thm.I.4.2].

We denote the space of square integrable martingales by \mathcal{M}^2 and its norm by $\|\cdot\|_{\mathcal{M}^2}$ with $\|M\|_{\mathcal{M}^2} = \mathbb{E}(M_T^2)^{1/2}$. We recall [HWY92, Thm.10.9.4] that the subspace of $\text{BMO}(\mathbb{P})$ -martingales $\text{BMO}(\mathbb{P})$ contains any square integrable martingale M with uniformly bounded jumps and bounded conditional expectations for increments of the quadratic variation process:

$$\sup_{0 \leq t \leq T} \left\| \mathbb{E}((M_T - M_t)^2 | \mathcal{F}_t) \right\|_{L^\infty(\mathbb{P})} = \sup_{0 \leq t \leq T} \left\| \mathbb{E}(\langle M \rangle_T - \langle M \rangle_t | \mathcal{F}_t) \right\|_{L^\infty(\mathbb{P})} \leq \text{const} < \infty.$$

We will assume that the continuous martingale B and the compensated measure $\tilde{\mu}$ of an integer-valued random measure μ (or $\tilde{\mu}$ alone, see Example 2.1.1 and Corollary 4.12 with trivial $B = 0$) jointly have the weak predictable representation property (weak PRP) w.r.t. the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, in that every square integrable martingale M has a (unique) representation, i.e.

$$\text{for all } M \in \mathcal{M}^2 \text{ there exists } Z, U \text{ such that } M = M_0 + \int Z dB + U * \tilde{\mu}, \quad (2.2)$$

with (unique) $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\tilde{\mu})$. Let us note that in the literature [JS03, III.§4c] or [HWY92, XIII.§2] the weak representation property is defined as a decomposition like (2.2) for any local martingale M with integrands Z, U being integrable in the sense of local martingales. Such clearly implies our formulation above. Indeed, for a (locally) square integrable martingale M in such a decomposition both integrands must be at least locally square integrable and $\langle M \rangle = \int |Z|^2 dt + |U|^2 * \nu$ by strong orthogonality of the stochastic integrals. Then $E[\langle M \rangle_T] < \infty$ implies that Z, U are in the respective \mathcal{L}^2 -spaces. We exemplify how (2.2) connects with a wide literature.

Example 2.1. *The weak predictable representation property (2.2) holds in the cases below. Cases 1.-4. are well known from classical theory [HWY92] (for details cf. [Bec06, Example 2.1]).*

1. *Let X be a Lévy process with $X_0 = 0$ and predictable characteristics (α, β, ν) (under \mathbb{P}). Then the continuous martingale part X^c (rescaled to a Brownian motion if $\beta \neq 0$, or being trivial if $\beta = 0$) and the compensated jump measure $\tilde{\mu}^X = \mu^X - \nu$ of X have the weak PRP w.r.t. the usual filtration \mathbb{F}^X generated by X . An example for a Lévy process of infinite activity is the Gamma process. One can add that weak PRP even holds in the sense of Thm III.4.34 from [JS03] for the more general class of PII-processes with independent increments. This class encompasses the more familiar Lévy processes without requiring time-homogeneity or stochastic continuity.*
2. *Assume that B and $\tilde{\mu}$ satisfy (2.2) under \mathbb{P} . Let \mathbb{P}' be an equivalent probability measure with density process Z . Then the Brownian motion $B' := B - \int (Z_-)^{-1} d\langle Z, B \rangle$ and $\tilde{\mu}' := \mu - \nu^{\mathbb{P}'}$ have the weak PRP (2.2) also w.r.t. \mathbb{P}' under the same filtration.*
3. *Let B be a Brownian motion independent of a step process X (in the sense of [HWY92, Ch. 11]). Then B and $\tilde{\mu}$, the compensated measure of the jump measure μ^X of X , have the weak PRP w.r.t. the usual filtration generated by X and B . An example for a step process is a multivariate (non-explosive) point process, as appearing in [CFJ16].*
4. *A (semi-)Markov chain X , possibly time-inhomogeneous, is a step process. Thus weak PRP (2.2) holds for a filtration generated by a Brownian motion and an independent Markov chain, relating later results to literature [CE10, CF14] on BSDEs driven by pure-jump Markov processes. Markov chains X on countable state spaces can be chosen [CE10] to take values in the set of unit vectors $\{e_i : i \in \mathbb{N}\}$ of the sequence space $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$, with jumps ΔX taking values $e_i - e_j$, $i, j \in \mathbb{N}$.*
5. *Note that in suitable cases, the pure jump martingale $U * \tilde{\mu}$ (for $U \in \mathcal{L}^2(\tilde{\mu})$) can be written as a series of mutually orthogonal martingales. More precisely, assume that the compensator coincides with the product measure $\lambda \otimes dt$, i.e. $\zeta = 1$. Let $(u^n)_{n \in \mathbb{N}}$ be an orthonormal basis (ONB) of the separable Hilbert space $L^2(\lambda)$ with scalar product $\langle u, v \rangle := \int_E u(e)v(e) \lambda(de)$. Let $U_t = \sum_{n \in \mathbb{N}} \langle U_t, u^n \rangle u^n$ be the basis expansion of U_t for $U \in \mathcal{L}^2(\tilde{\mu})$, $t \leq T$. Then it holds (in \mathcal{M}^2)*

$$U * \tilde{\mu} = \sum_{n \in \mathbb{N}} \int_0^T \langle U_t, u^n \rangle \int_E u^n(e) \tilde{\mu}(dt, de) =: \sum_{n \in \mathbb{N}} \int_0^T \alpha_t^n dL_t^n = \sum_{n \in \mathbb{N}} \alpha^n \cdot L^n, \quad (2.3)$$

for $\alpha_t^n := \langle U_t, u^n \rangle$ and $L^n := u^n * \tilde{\mu}$. Indeed, setting $F_t^n := \sum_{k=1}^n \langle U_t, u^k \rangle u^k = \sum_{k=1}^n \alpha_t^k u^k$ one sees that $\|\sum_{k=1}^\infty |\alpha^k|^2\|_{L^1(\mathbb{P} \otimes dt)} \leq \|U\|_{\mathcal{L}^2(\tilde{\mu})}^2 < \infty$. By dominated convergence one obtains as $n \rightarrow \infty$

$$\|F^n - U\|_{\mathcal{L}^2(\tilde{\mu})}^2 = \mathbb{E} \left(\int_0^T \int_E |F_t^n(e) - U_t(e)|^2 \lambda(de) dt \right) = \mathbb{E} \left(\int_0^T \sum_{k=n+1}^\infty |\alpha_t^k|^2 dt \right) \rightarrow 0.$$

Isometry implies that the stochastic integrals $F^n * \tilde{\mu}$ converge to $U * \tilde{\mu}$ in \mathcal{M}^2 , proving (2.3).

In particular, we see how the PRP (2.2) w.r.t. a random measure can be rewritten as series of ordinary stochastic integrals w.r.t. scalar-valued strongly orthogonal martingales L^n , which are in fact Lévy processes with deterministic characteristics $(0, 0, \int u^n(e) \lambda(de))$. In this sense, the general condition (2.2) links well with results on PRP and BSDEs for Lévy processes in [NS00, NS01] who study a specific Teugels martingale basis consisting of compensated power jump processes for Lévy processes which satisfy exponential moment conditions. For a systematic analysis of related PRP results, comprising general Lévy processes, see [DTE15b, DTE15a].

6. Note that the previous arguments extend to the general case with $\zeta \neq 1$ in (2.1), letting U^n be in $\mathcal{L}^2(\tilde{\mu})$ such that for all $t \leq T$ the sequence $(U_t^n)_{n \in \mathbb{N}}$ is ONB of $L^2(\lambda_t)$ for $d\lambda_t = \zeta_t d\lambda$ with scalar product $\langle u, v \rangle_t := \int_E u(e)v(e)\zeta(t, e)\lambda(de)$. Analogously to case 5. above, with $\alpha_t^n := \langle U_t, U_t^n \rangle_t$ and $L^n := U^n * \tilde{\mu}$ one gets equalities of martingales (in \mathcal{M}^2)

$$U * \tilde{\mu} = \sum_{n \in \mathbb{N}} \int_0^T \langle U_t, U_t^n \rangle_t \int_E U_t^n(e) \tilde{\mu}(dt, de) =: \sum_{n \in \mathbb{N}} \alpha^n \cdot L^n.$$

To proceed, we now define a solution of the Backward SDE with jumps to be a triple (Y, Z, U) of processes in the space $\mathcal{S}^p \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ for a suitable $p \in (1, \infty]$ that satisfies

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \quad (2.4)$$

for given data (ξ, f) , consisting of a \mathcal{F}_T -measurable random variable ξ and a generator function $f_t(y, z, u) = f(\omega, t, y, z, u)$. The values p will be specified below in the respective results, although a particular focus will be on bounded BSDE solutions (i.e. $p = \infty$). Because we permit ν to be time-inhomogeneous with a bounded but possibly non-constant density ζ in (2.1), it does not hold in general that U_t is a.e. in $L^2(\lambda)$ for $U \in \mathcal{L}^2(\tilde{\mu})$. This requires some extra consideration about the domain of definition and measurability of f , as the generator function f needs to be defined for u -arguments from a suitable domain, which cannot be some fixed L^2 -space in general (and needs to be larger than $L^2(\lambda)$), as integrability of $u = U_t(\omega, \cdot)$ over $e \in E$ may vary with (ω, t) . On suitable larger domains, one typically may have to admit for f to attain non-finite values. To this end, let us denote by $L^0(\mathcal{B}(E), \lambda)$ the space of all $\mathcal{B}(E)$ -measurable functions with the topology of convergence in measure and define

$$|u - u'|_t := \left(\int_E |u(e) - u'(e)|^2 \zeta(t, e) \lambda(de) \right)^{\frac{1}{2}}, \quad (2.5)$$

for functions u, u' in $L^0(\mathcal{B}(E), \lambda)$. Terminal conditions ξ for BSDE considered in this paper will be taken to be square integrable $\xi \in L^2(\mathcal{F}_T)$ and often even as bounded $\xi \in L^\infty(\mathcal{F}_T)$. Generator functions $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^0(\mathcal{B}(E), \lambda) \rightarrow \mathbb{R}$ are always taken to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{B}(E), \lambda))$ -measurable. Main results such as Theorems 3.9, 4.3 and 4.13 will be stated for families of generators having the form

$$f_t(y, z, u) := \hat{f}_t(y, z) + \int_A g_t(y, z, u(e), e) \zeta(t, e) \lambda(de) \quad (\text{where finitely defined}) \quad (2.6)$$

and $f_t(y, z, u) := \infty$ elsewhere, or more specially (for a g -component not depending on y, z)

$$f_t(y, z, u) := \hat{f}_t(y, z) + \int_A g_t(u(e), e) \zeta(t, e) \lambda(de) \quad (\text{where finitely defined}) \quad (2.7)$$

and $f_t(y, z, u) := \infty$ elsewhere, for a $\mathcal{B}(E)$ -measurable set A and component functions \hat{f}, g where $\hat{f} : \Omega \times [0, T] \times \mathbb{R}^{1+d} \rightarrow \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})$ -measurable and $g : \Omega \times [0, T] \times \mathbb{R}^{1+d} \times \mathbb{R} \times E \rightarrow \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+2}) \otimes \mathcal{B}(E)$ -measurable. Clearly statements for generators of the form (2.6) are also true for those of the (more particular) form (2.7). (In)finite activity relates to generators with $\lambda(A) < \infty$ (respectively $\lambda(A) = \infty$). A simple but useful technical Lemma clarifies how we can (and always will) choose a bounded representative for U in a BSDE solution (Y, Z, U) with bounded Y .

Lemma 2.2. *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ be a solution of some JBSDE (2.4) with data (ξ, f) . Then there exists a representative U' of U , bounded pointwise by $2|Y|_\infty$, such that $U' = U$ in $\mathcal{L}^2(\tilde{\mu})$ and $\mathbb{P} \otimes dt$ -a.e., and (Y, Z, U') solves the BSDE (ξ, f) .*

Proof. We reproduce a brief argument sufficient to our general setting, similarly to e.g. [Mor09, Cor.1] or [Bec06, proof of Thm.3.5]. Use that $\mu(\omega, dt, de) = \sum_{s \geq 0} \mathbb{1}_D(\omega, s) \delta_{(s, \beta_s(\omega))}(dt, de)$ for an optional E -valued process β and a thin set D , since μ is an integer-valued random measure [JS03, II.§1b]. Clearly the jump $\Delta Y_t(\omega) = (Y_t - Y_{t-})(\omega) = \int_E U_t(\omega, e) \mu(\omega; \{t\}, de)$ is equal to $\mathbb{1}_D(\omega, t) U_t(\omega, \beta_t(\omega))$ and bounded by $2|Y|_\infty$. For $U'_t(\omega, e) := U_t(\omega, e) \mathbb{1}_D(\omega, t) \mathbb{1}_{\{\beta_t\}}(e)$, we have $U_t(\omega, \beta_t(\omega)) = U'_t(\omega, \beta_t(\omega))$ on D , and $\sum_{s \geq 0} \mathbb{1}_D(\omega, s) |U_s - U'_s|^2(\omega, \beta_s(\omega)) = 0$ implies $E[|U - U'|^2 * \nu_T] = E[|U - U'|^2 * \mu_T] = 0$. Since $U = U'$ in $\mathcal{L}^2(\tilde{\mu})$ and $U_t = U'_t$ in $L^0(\mathcal{B}(E), \lambda)$, the BSDE is solved by (Y, Z, U') . \square

Under these conditions, we can and will take U to be bounded by twice the norm of Y ; Defining $|U|_\infty := \text{ess sup}_{(\omega, t, e)} |U_t(e)|$ for $U \in \mathcal{L}^2(\tilde{\mu})$ yields $|U|_\infty \leq 2|Y|_\infty$ for any bounded BSDE solution (Y, Z, U) . The next lemma notes that the stochastic integrals of bounded JBSDE solutions are BMO-martingales when some truncated generator function is bounded from above (below) by $+(-)\langle M \rangle$ for a BMO-martingale M ; Moreover, their BMO-norms depend only on $|Y|_\infty$, the BMO-norm of M and the horizon T . See [Ken15, Lem.1.3] for details of the proof, and note that BMO-properties of integrals of (bounded) BSDEs are of course a well-studied topic, cf. [MC14] and references therein.

Lemma 2.3. *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ be a bounded solution to the BSDE (ξ, f) . Assume there is $M \in \text{BMO}(\mathbb{P})$ such that $\int_t^T f_s(Y_{s-}, Z_s, U_s) ds \leq \langle M \rangle_T - \langle M \rangle_t$ or $-\int_t^T f_s(Y_{s-}, Z_s, U_s) ds \leq \langle M \rangle_T - \langle M \rangle_t$. Then $\int Z dB$ and $U * \tilde{\mu}$ are BMO-martingales and their BMO-norms (resp. L^2 -norms) are bounded by a constant depending on $|Y|_\infty$ and $\|M\|_{\text{BMO}(\mathbb{P})}$ (resp. on $|Y|_\infty$, $\|M\|_{\mathcal{M}^2}$).*

3 Comparison theorems and a-priori-estimates

The next proposition sets the stage for the main comparison Theorem 3.9 and the a-priori- L^∞ -estimate Theorem 3.11 of this section. As usual for BSDE comparison results, the proof relies on a linearization technique and a change of measure argument. In a framework with random measures, it is very close to the seminal Thm. 2.5 from [Roy06] with slight generalizations that are needed in the sequel. While we follow her line of proof, some details for the change of measure argument are elaborated slightly differently, measurable dependencies of the random field γ are specified in more detail, and we assume less on the generators. Instead of imposing specific conditions on the generators which imply existence of solutions, we only insist that we have solutions and impose a generalized (\mathbf{A}_γ) -condition as explained in Example 3.8.1.

Proposition 3.1. *Let $(Y^i, Z^i, U^i) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ be solutions to the BSDE (2.4) for data (ξ_i, f_i) , $i = 1, 2$. Assume that f_2 is Lipschitz continuous w.r.t. y and z . Let $\gamma : \Omega \times [0, T] \times \mathbb{R}^{d+3} \times E \rightarrow [-1, \infty)$ with $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y, z, u, u'}(e)$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$ -measurable function such that for $\bar{\gamma} := \gamma^{Y^2, Z^2, U^1, U^2}$ it holds*

$$f_2(t, Y_{t-}^2, Z_{t-}^2, U_t^1) - f_2(t, Y_{t-}^2, Z_{t-}^2, U_t^2) \leq \int_E \bar{\gamma}_t(e) (U_t^1(e) - U_t^2(e)) \zeta(t, e) \lambda(de), \quad \mathbb{P} \otimes dt\text{-a.e.} \quad (3.1)$$

*and the stochastic exponential $\mathcal{E}(\int \beta dB + \bar{\gamma} * \tilde{\mu})$ is a martingale for β from (3.2).*

Then a comparison result holds, that is $\xi_1 \leq \xi_2$ and $f_1(t, Y_{t-}^1, Z_{t-}^1, U_t^1) \leq f_2(t, Y_{t-}^1, Z_{t-}^1, U_t^1)$ together imply $Y_t^1 \leq Y_t^2$ for all $t \leq T$.

Proof. We define $\hat{\xi} := \xi_1 - \xi_2$, $\hat{Y} := Y^1 - Y^2$, $\hat{Z} := Z^1 - Z^2$ and $\hat{U} := U^1 - U^2$. The processes

$$\begin{aligned} \alpha_s &:= \mathbb{1}_{\{Y_{s-}^1 \neq Y_{s-}^2\}} \frac{f_2(s, Y_{s-}^1, Z_{s-}^1, U_s^1) - f_2(s, Y_{s-}^2, Z_{s-}^1, U_s^1)}{(Y_{s-}^1 - Y_{s-}^2)}, \\ \beta_s &:= \mathbb{1}_{\{Z_s^1 \neq Z_s^2\}} \frac{f_2(s, Y_{s-}^2, Z_s^1, U_s^1) - f_2(s, Y_{s-}^2, Z_s^2, U_s^1)}{\|Z_s^1 - Z_s^2\|^2} (Z_s^1 - Z_s^2) \end{aligned} \quad (3.2)$$

and $R_t := \exp(\int_0^t \alpha_s ds)$ are bounded due to the Lipschitz assumption on f_2 . As in [Roy06], applying Itô's formula to $R\hat{Y}$ between $\tau \wedge t$ and $\tau \wedge T$ for some stopping times τ yields

$$\begin{aligned} (R\hat{Y})_{\tau \wedge t} &= (R\hat{Y})_{\tau \wedge T} + \int_{\tau \wedge t}^{\tau \wedge T} R_s (f_1(s, Y_{s-}^1, Z_s^1, U_s^1) - f_2(s, Y_{s-}^2, Z_s^2, U_s^2)) ds \\ &\quad - \int_{\tau \wedge t}^{\tau \wedge T} R_s \hat{Z}_s dB_s - \int_{\tau \wedge t}^{\tau \wedge T} \int_E R_s \hat{U}_s(e) \tilde{\mu}(ds, de) - \int_{\tau \wedge t}^{\tau \wedge T} R_s \alpha_s \hat{Y}_{s-} ds. \end{aligned}$$

Set $M := \int R \hat{Z} dB + (R\hat{Y}) * \tilde{\mu}$ and $N := \int \beta dB + \bar{\gamma} * \tilde{\mu}$. Then $d\mathbb{Q} := \mathcal{E}(N)_T d\mathbb{P}$ defines an absolutely continuous probability by the martingale property of the stochastic exponential $\mathcal{E}(N) \geq 0$; cf. [HWY92, Lem.9.40]. By Girsanov $L := M - \langle M, N \rangle$ is a local \mathbb{Q} -martingale, and the inequality

$$\begin{aligned} f_1(s, Y_{s-}^1, Z_s^1, U_s^1) - f_2(s, Y_{s-}^2, Z_s^2, U_s^2) &\leq \alpha_s \hat{Y}_{s-} + \beta_s \hat{Z}_s + \int_E \bar{\gamma}_s(e) \hat{U}_s(e) \zeta_s(e) \lambda(de) \mathbb{P} \otimes ds\text{-a.e.} \\ \text{implies } (R\hat{Y})_{\tau \wedge t} &\leq (R\hat{Y})_{\tau \wedge T} - (L_T^\tau - L_t^\tau). \end{aligned} \quad (3.3)$$

Localizing L along a sequence of stopping times $\tau_n \uparrow \infty$ and taking conditional expectations, we obtain $\mathbb{E}_{\mathbb{Q}}((R\hat{Y})_{t \wedge \tau_n} | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{Q}}((R\hat{Y})_{\tau_n \wedge T} | \mathcal{F}_t)$ for each $n \in \mathbb{N}$. Dominated convergence yields the estimate $R_t \hat{Y}_t \leq \mathbb{E}_{\mathbb{Q}}(R_T \hat{\xi} | \mathcal{F}_t) \leq 0$ and thus $Y_t^1 \leq Y_t^2$. \square

Remark 3.2. 1. Switching roles of f_1 and f_2 , one gets that if f_1 is Lipschitz in y, z and satisfies (3.1) instead of f_2 , then $\xi_1 \leq \xi_2$ and $f_1(t, Y_{t-}^2, Z_t^2, U_t^2) \leq f_2(t, Y_{t-}^2, Z_t^2, U_t^2)$ imply $Y_t^1 \leq Y_t^2$.
2. The result of Proposition 3.1 remains valid (with a similar proof) if one requires that the Y -components of JBSDE solutions to compare are in \mathcal{S}^2 instead of \mathcal{S}^∞ , and the stochastic exponential $\mathcal{E}(\beta \bullet W + \bar{\gamma} * \tilde{\mu})$ is in \mathcal{S}^2 . However, as it is stated in the present form, Proposition 3.1 is exactly what we will need to apply in the sequel to derive main result such as Theorems 4.3 and 4.13.

Example 3.3. Sufficient conditions for $\mathcal{E}(\bar{\gamma} * \tilde{\mu})$ to be a martingale are, for instance,

1. $\Delta(\bar{\gamma} * \tilde{\mu}) > -1$ and $\mathbb{E}(\exp(\langle \bar{\gamma} * \tilde{\mu} \rangle_T)) = \mathbb{E}(\exp(\int_0^T \int_E |\bar{\gamma}_s(e)|^2 \nu(ds, de))) < \infty$; see [PS08, Thm.9]. This holds i.p. if $\int_E |\bar{\gamma}_s(e)|^2 \zeta(s, e) \lambda(de) < \text{const.} < \infty \mathbb{P} \otimes ds\text{-a.e.}$ and $\bar{\gamma} > -1$.
2. $\Delta(\bar{\gamma} * \tilde{\mu}) \geq -1 + \delta$ for $\delta > 0$ and $\bar{\gamma} * \tilde{\mu}$ is a BMO(\mathbb{P})-martingale due to Kazamaki [Kaz79].
3. $\Delta(\bar{\gamma} * \tilde{\mu}) \geq -1$ and $\bar{\gamma} * \tilde{\mu}$ is a uniformly integrable martingale and $\mathbb{E}(\exp(\langle \bar{\gamma} * \tilde{\mu} \rangle_T)) < \infty$; see [LM78, Thm.I.8]. Such a condition is satisfied when $\bar{\gamma}$ is bounded and $|\bar{\gamma}| \leq \psi$, $\mathbb{P} \otimes dt \otimes \lambda\text{-a.e.}$ for a function $\psi \in L^2(\lambda)$ and $\zeta \equiv 1$. The latter is what is required for instance in the comparison Thm.4.2 of [QS13].

Note that under above conditions, also the stochastic exponential $\mathcal{E}(\int \beta dB + \bar{\gamma} * \tilde{\mu})$ for β bounded and predictable is a martingale, as it is easily seen by Novikov's criterion.

In the statement of Proposition 3.1, the dependence of the process $\bar{\gamma}$ on the BSDE solutions is not needed for the proof as the same result holds if $\bar{\gamma}$ is just a predictable process such that the estimate on the generator f_2 and the martingale property (3.1) hold. The further functional dependence is needed for the sequel, as required in the following

Definition 3.4. We say that an \mathbb{R} -valued generator function f satisfies condition (\mathbf{A}_γ) if there is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$ -measurable function $\gamma : \Omega \times [0, T] \times \mathbb{R}^{d+3} \times E \rightarrow (-1, \infty)$ given by $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y, z, u, u'}(e)$ such that for all $(Y, Z, U, U') \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times (\mathcal{L}^2(\tilde{\mu}))^2$ with $|U|_\infty < \infty$, $|U'|_\infty < \infty$ it holds for $\bar{\gamma} := \gamma^{Y, Z, U, U'}$

$$\begin{aligned} f_t(Y_{t-}, Z_t, U_t) - f_t(Y_{t-}, Z_t, U'_t) &\leq \int_E \bar{\gamma}_t(e) (U_t(e) - U'_t(e)) \zeta(t, e) \lambda(de), \mathbb{P} \otimes dt\text{-a.e.} \\ \text{and } \mathcal{E}(\int \beta dB + \bar{\gamma} * \tilde{\mu}) &\text{ is a martingale for every bounded and predictable } \beta. \end{aligned} \quad (3.4)$$

We will say that f satisfies condition (\mathbf{A}'_γ) if the above holds for all bounded U and U' with additionally $U * \tilde{\mu}$ and $U' * \tilde{\mu}$ in BMO(\mathbb{P}).

Clearly, existence and applicability of a suitable comparison result for solutions to JBSDEs implies their uniqueness. In other words assuming there exists a bounded solution for a driver Lipschitz w.r.t. y and z which satisfies (\mathbf{A}_γ) or (\mathbf{A}'_γ) , we obtain that such a solution is unique.

Example 3.5. The natural candidate for γ for drivers f of the form (2.6) is given by

$$\gamma_s^{y,z,u,u'}(e) = \frac{g_s(y,z,u,e) - g_s(y,z,u',e)}{u - u'} \mathbb{1}_A(e) \mathbb{1}_{\{u \neq u'\}}, \quad (3.5)$$

which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$ -measurable since g is. Assuming absolute continuity of g in u , we can express $\gamma_s^{y,z,u,u'}(e) = \int_0^1 \frac{\partial}{\partial u} g_s(y,z,tu + (1-t)u',e) dt \mathbb{1}_A(e)$, by noting that

$$(u - u') \int_0^1 \frac{\partial}{\partial u} g_s(y,z,tu + (1-t)u',e) dt \mathbb{1}_A(e) = \int_0^1 \frac{\partial}{\partial t} [g_s(y,z,tu + (1-t)u',e)] dt \mathbb{1}_A(e).$$

For generators of type (2.7) the γ simply is $\gamma_s^{y,z,u,u'}(e) = \int_0^1 \frac{\partial}{\partial u} g_s(tu + (1-t)u',e) dt \mathbb{1}_A(e)$.

Definition 3.6. We say that a generator f satisfies condition $(\mathbf{A}_{\text{fin}})$ or $(\mathbf{A}_{\text{inf}})$ (on a set D) if

1. $(\mathbf{A}_{\text{fin}})$: f is of the form (2.6) with $\lambda(A) < \infty$, is Lipschitz continuous w.r.t. y and z uniformly in (t, ω, u) , and the map $u \mapsto g(t, y, z, u, e)$ is absolutely continuous (in u) for all (ω, t, y, z, e) (in $D \subseteq \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times E$), i.e. $g(t, y, z, u, e) = g(0) + \int_0^u g'(t, y, z, x, e) dx$, with density function g' being strictly greater than -1 (on D) and locally bounded (in u) from above, uniformly in (ω, t, y, z, e) .
2. $(\mathbf{A}_{\text{inf}})$: f is of the form (2.7), is Lipschitz continuous w.r.t. y and z uniformly in (t, ω, u) , and the map $u \mapsto g_t(u, e)$ is absolutely continuous (in u) for all (ω, t, e) (in D), i.e. $g(t, u, e) = g(0) + \int_0^u g'(t, x, e) dx$, with density function g' being such that for all $c \in (0, \infty)$ there exists $K(c) \in \mathbb{R}$ and $\delta(c) \in (0, 1)$ with $-1 + \delta(c) \leq g'(x)$ and $|g'(x)| \leq K(c)|x|$ for all x with $|x| \leq c$.

Remark 3.7. Note that under condition $(\mathbf{A}_{\text{inf}})$ the density function g' is necessarily locally bounded, in particular with $|g'(x)| \leq K(c)c =: \bar{K}(c) < \infty$ for all $x \in [-c, c]$.

Example 3.8. Sufficient conditions for condition (\mathbf{A}_γ) and (\mathbf{A}'_γ) are

1. γ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$ -measurable function satisfying the inequality in (3.4) and

$$C_1(1 \wedge |e|) \leq \gamma_t^{y,z,u,u'}(e) \leq C_2(1 \vee |e|)$$

on $E = \mathbb{R}^l \setminus \{0\}$ ($l \in \mathbb{N}$), for some $C_1 \in (-1, 0]$ and $C_2 > 0$. In this case $\exp(\langle \int \beta dB + \bar{\gamma} * \tilde{\mu} \rangle_T)$ is clearly bounded and the jumps of $\int \beta dB + \bar{\gamma} * \tilde{\mu}$ are bigger than -1 . Hence $\mathcal{E}(\int \beta dB + \bar{\gamma} * \tilde{\mu})$ is a positive martingale [PS08, Thm.9]. Thus Definition 3.4 generalizes the original (\mathbf{A}_γ) -condition introduced by [Roy06] for Poisson random measures.

2. $(\mathbf{A}_{\text{fin}})$ is sufficient for (\mathbf{A}_γ) . This follows from Example 3.3.1, (3.5) and $\lambda(A) < \infty$.
3. $(\mathbf{A}_{\text{inf}})$ is sufficient for (\mathbf{A}'_γ) . To see this, let u, u' be bounded by c and γ be the natural candidate in Example 3.5. Then $|\gamma_s^{y,z,u,u'}(e)| \leq \int_{u'}^u |g'(x)| dx / (u - u') \leq K(c)(|u| + |u'|)$. Hence $\int \beta dB + \bar{\gamma} * \tilde{\mu}$ is a BMO-martingale by the BMO-property of $U * \tilde{\mu}$ and $U' * \tilde{\mu}$ with some lower bound $-1 + \delta$ for its jumps. And $\mathcal{E}(\int \beta dB + \bar{\gamma} * \tilde{\mu})$ is a martingale by part 2 of Example 3.3.
4. Condition $(\mathbf{A}_{\text{fin}})$ above is satisfied if, e.g., f is of the form (2.6) with $\lambda(A) < \infty$, is Lipschitz continuous w.r.t. y and z , and the map $u \mapsto g(t, y, z, u, e)$ is continuously differentiable for all (ω, t, y, z, e) (in D) such that the derivative is strictly greater than -1 (on $D \subseteq \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times E$) and locally bounded (in u) from above, uniformly in (ω, t, y, z, e) .
5. Condition $(\mathbf{A}_{\text{inf}})$ is valid if for instance f is of the form (2.7), is Lipschitz continuous w.r.t. y and z , and the map $u \mapsto g_t(u, e)$ is twice continuously differentiable for all (ω, t, e) with the derivatives being locally bounded uniformly in (ω, t, e) , the first derivative being (locally) bounded away from -1 with a lower bound $-1 + \delta$ for some $\delta > 0$, and $\frac{\partial g}{\partial u}(t, 0, e) \equiv 0$.

As an application of the above, we can now provide simple conditions for comparison in terms of concrete properties of the generator function, which are easier to verify than the more general but abstract conditions on the existence of a suitable function γ as in Proposition 3.1 or the general conditions by [CE10]. Note that no convexity is required in the z or u argument of the generator. The result will be applied later to prove existence and uniqueness of JBSDE solutions.

Theorem 3.9 (Comparison Theorem). *A comparison result between bounded BSDE solutions in the sense of Proposition 3.1 holds true in each of the following cases:*

1. (finite activity) f_2 satisfies $(\mathbf{A}_{\text{fin}})$.
2. (infinite activity) f_2 satisfies $(\mathbf{A}_{\text{inf}})$ and $U^1 * \tilde{\mu}$ and $U^2 * \tilde{\mu}$ are $\text{BMO}(\mathbb{P})$ -martingales for the corresponding JBSDE solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) .

Proof. This follows directly from Proposition 3.1 and Example 3.8, noting that representation (3.5) in connection with condition $(\mathbf{A}_{\text{fin}})$ resp. $(\mathbf{A}_{\text{inf}})$ meets the sufficient conditions in Example 3.3. \square

Unlike classical a-priori estimates that offer some L^2 -norm estimates for the BSDE solution in terms of the data, the next result gives a simple L^∞ -estimate for the Y -component of the solution. Such will be useful for the derivation of BSDE solution bounds and for truncation arguments.

Proposition 3.10. *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ be a solution to the BSDE (ξ, f) with $\xi \in L^\infty(\mathcal{F}_T)$, f be Lipschitz continuous w.r.t. (y, z) with Lipschitz constant $K_f^{y,z}$ and satisfying (\mathbf{A}_γ) with $f.(0, 0, 0)$ bounded. Then $|Y_t| \leq \exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f.(0, 0, 0)|_\infty)$ for $t \leq T$.*

Proof. Set $(Y^1, Z^1, U^1) = (Y, Z, U)$, $(\xi^1, f^1) = (\xi, f)$, $(Y^2, Z^2, U^2) = (0, 0, 0)$ and $(\xi^2, f^2) = (0, f)$. Then following the proof of Proposition 3.1, equation (3.3) becomes

$$(RY)_{\tau \wedge t} \leq (RY)_{\tau \wedge T} + \int_{\tau \wedge t}^{\tau \wedge T} R_s f_s(0, 0, 0) ds - (L_T^\tau - L_t^\tau), \quad t \in [0, T],$$

for all stopping times τ where $L := M - \langle M, N \rangle$ is in $\mathcal{M}_{\text{loc}}(\mathbb{Q})$, $M := \int RZ dB + (RU) * \tilde{\mu}$ is in \mathcal{M}^2 , $N := \int \beta dB + \bar{\gamma} * \tilde{\mu}$ with $\bar{\gamma} := \gamma^{0,0,U,0}$ and the probability measure $\mathbb{Q} \approx \mathbb{P}$ is given by $d\mathbb{Q} := \mathcal{E}(N)_T d\mathbb{P}$. Localizing L along some sequence $\tau^n \uparrow \infty$ of stopping times yields $\mathbb{E}_{\mathbb{Q}}((RY)_{\tau^n \wedge t} | \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{Q}}((RY)_{\tau^n \wedge T} + \int_{\tau^n \wedge t}^{\tau^n \wedge T} R_s f_s(0, 0, 0) ds | \mathcal{F}_t)$. By dominated convergence, we conclude that \mathbb{P} -a.e

$$Y_t \leq \mathbb{E}_{\mathbb{Q}}\left(\frac{R_T}{R_t} \xi + \int_t^T \frac{R_s}{R_t} f_s(0, 0, 0) ds \mid \mathcal{F}_t\right) \leq e^{K_f^{y,z}(T-t)}(|\xi|_\infty + (T-t)|f.(0, 0, 0)|_\infty).$$

Analogously, if we define $\bar{N} := \int \beta dB + \bar{\gamma} * \tilde{\mu}$ with $\bar{\gamma} := \gamma^{0,0,0,U}$, and $\bar{\mathbb{Q}}$ equivalent to \mathbb{P} via $d\bar{\mathbb{Q}} := \mathcal{E}(\bar{N})_T d\mathbb{P}$, we deduce that $\bar{L} := M - \langle M, \bar{N} \rangle$ is in $\mathcal{M}_{\text{loc}}(\bar{\mathbb{Q}})$ and

$$(RY)_{\tau \wedge t} \geq (RY)_{\tau \wedge T} + \int_{\tau \wedge t}^{\tau \wedge T} R_s f_s(0, 0, 0) ds - (\bar{L}_T^\tau - \bar{L}_t^\tau), \quad t \in [0, T],$$

for all stopping times τ . This yields the required lower bound. \square

Again, we can specify explicit conditions on the generator function that are sufficient to ensure the more abstract assumptions of the previous result.

Theorem 3.11. *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ be a solution to the BSDE (ξ, f) with ξ in $L^\infty(\mathcal{F}_T)$, f being Lipschitz continuous w.r.t. (y, z) with Lipschitz constant $K_f^{y,z}$ such that $f.(0, 0, 0)$ is bounded. Assume that one of the following conditions holds:*

1. (finite activity) f satisfies $(\mathbf{A}_{\text{fin}})$.
2. (infinite activity) f satisfies $(\mathbf{A}_{\text{inf}})$ and $U * \tilde{\mu}$ is a $\text{BMO}(\mathbb{P})$ -martingale.

Then $|Y_t| \leq \exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f_s(0, 0, 0)|_\infty)$ holds for all $t \leq T$, in particular $|Y|_\infty \leq \exp(K_f^{y,z}T)(|\xi|_\infty + T|f_s(0, 0, 0)|_\infty)$.

Proof. This follows directly from Proposition 3.10 and Example 3.8, since f satisfies condition (\mathbf{A}_γ) (resp. (\mathbf{A}'_γ)) using equation (3.5). \square

In the last part of this section we apply our comparison theorem for more concrete generators. To this end, we consider a generator f being truncated at bounds $a < b$ (depending on time only) as

$$\tilde{f}_t(y, z, u) := f_t(\kappa(t, y), z, \kappa(t, y + u) - \kappa(t, y)), \quad (3.6)$$

with $\kappa(t, y) := (a(t) \vee y) \wedge b(t)$. Next, we show that if a generator satisfies (\mathbf{A}_γ) within the truncation bounds, then the truncated generator satisfies (\mathbf{A}_γ) everywhere.

Lemma 3.12. *Let f satisfy (3.4) for Y, U such that $a(t) \leq Y_{t-}, Y_{t-} + U_t(e), Y_{t-} + U'_t(e) \leq b(t)$, $t \in [0, T]$ and let γ satisfy one of the conditions of Example 3.3 for the martingale property of $\mathcal{E}(\bar{\gamma} * \tilde{\mu})$. Then \tilde{f} satisfies (3.4). Especially, if f satisfies $(\mathbf{A}_{\text{fin}})$ on the set where $a(t) \leq y, y + u \leq b(t)$ then \tilde{f} is Lipschitz in (y, z) , locally Lipschitz in u and satisfies (\mathbf{A}_γ) .*

Proof. Using monotonicity of $x \mapsto \kappa(t, x)$, we get that $\tilde{f}_t(Y_{t-}, Z_t, U_t) - \tilde{f}_t(Y_{t-}, Z_t, U'_t)$ equals

$$\begin{aligned} & f_t(\kappa(t, Y_{t-}), Z_t, \kappa(t, Y_{t-} + U_t) - \kappa(t, Y_{t-})) - f_t(\kappa(t, Y_{t-}), Z_t, \kappa(t, Y_{t-} + U'_t) - \kappa(t, Y_{t-})) \\ & \leq \int_E \bar{\gamma}_t(e) (\kappa(t, Y_{t-} + U_t(e)) - \kappa(t, Y_{t-} + U'_t(e))) \zeta(t, e) \lambda(de) \\ & \leq \int_E \bar{\gamma}_t(e) (\mathbf{1}_{\{\bar{\gamma} \geq 0, U \geq U'\}} + \mathbf{1}_{\{\bar{\gamma} < 0, U < U'\}}) (U_t(e) - U'_t(e)) \zeta(t, e) \lambda(de). \end{aligned}$$

Setting $\bar{\gamma}^* := \bar{\gamma}(\mathbf{1}_{\{\bar{\gamma} \geq 0, U \geq U'\}} + \mathbf{1}_{\{\bar{\gamma} < 0, U < U'\}})$ we see that the stochastic exponential $\mathcal{E}(\int \beta dB + \bar{\gamma}^* * \tilde{\mu})$ is a martingale for all bounded and predictable processes β and \tilde{f} satisfies (3.4). The latter claim easily follows from the fact that if f satisfies $(\mathbf{A}_{\text{fin}})$ on $a(t) \leq y, y + u \leq b(t)$ then f satisfies (3.4) on $a(t) \leq Y_{t-}, Y_{t-} + U_t(e), Y_{t-} + U'_t(e) \leq b(t)$ using Example 3.8.2. The Lipschitz properties of \tilde{f} follow from the fact that κ is a contraction and f is Lipschitz within the truncation bounds. \square

Concrete L^∞ -bounds for bounded solutions to BSDE (ξ, f) with suitable \hat{f} -part are provided by

Proposition 3.13. *Let f be a generator of the form (2.6) with $|\hat{f}_t(y, z)| \leq K_1 + K_2|y|$ for some $K_1, K_2 \geq 0$, $g_t(y, z, 0, e) \equiv 0$ and $\xi \in L^\infty(\mathcal{F}_T)$ with $c_1 \leq \xi \leq c_2$ for some $c_1, c_2 \in \mathbb{R}$. Assume that there are solutions a and b to the ODEs $y'(t) = K_1 + K_2|y(t)|$, $y(T) = c_1$ and $y'(t) = -(K_1 + K_2|y(t)|)$, $y(T) = c_2$ respectively, such that $a \leq b$ on $[0, T]$. If the truncated generator \tilde{f} in (3.6) satisfies (\mathbf{A}_γ) and is Lipschitz in (y, z) , then any solution $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the JBSDE (ξ, \tilde{f}) also solves the JBSDE (ξ, f) and satisfies $a(t) \leq \tilde{Y}_t \leq b(t)$, $t \in [0, T]$.*

Proof. We set $Y_t := \kappa(t, \tilde{Y}_t)$, $Z_t := \tilde{Z}_t$, $U_t(e) := \kappa(t, \tilde{Y}_{t-} + \tilde{U}_t(e)) - \kappa(t, \tilde{Y}_{t-})$ and

$$f_t^i(y, z, u) := \hat{f}_t^i(\kappa(t, y), z) + \int_E g_t(\kappa(t, y), z, \kappa(t, y + u) - \kappa(t, y), e) \zeta(t, e) \lambda(de)$$

with $\hat{f}_t^1(y, z) := -(K_1 + K_2|y|)$, $\hat{f}_t^2(y, z) := \hat{f}_t(y, z)$ and $\hat{f}_t^3(y, z) := K_1 + K_2|y|$. By the assumptions on the ODEs, we have that $(a(t), 0, 0)$ solves the BSDE (c_1, f^1) and $(b(t), 0, 0)$ solves the BSDE (c_2, f^3) . Taking into account that $\tilde{f}^1 \leq \tilde{f}^2 \leq \tilde{f}^3$, $c_1 \leq \xi \leq c_2$ and \tilde{f}^2 satisfies (\mathbf{A}_γ) , comparison theorem Proposition 3.1 yields $a(t) \leq \tilde{Y}_t \leq b(t)$. Hence, Y and \tilde{Y} are indistinguishable, $U = \tilde{U}$ in $\mathcal{L}^2(\tilde{\mu})$ and $(\tilde{Y}, \tilde{Z}, \tilde{U})$ solves the BSDE (ξ, f) . \square

In the next section, we apply these results to two situations: Using Corollary 4.4, we give an alternative proof of Thm.3.5 of [Bec06] via a comparison principle instead of an argument with stopping times. Moreover, the estimates in Corollary 4.6 are applied to solve the power utility maximization problem via a JBSDE approach in Section 5.2.

4 Existence and Uniqueness of bounded solutions

This section studies BSDE with jumps by the monotone stability approach. Building on results for finite activity (obtained very straightforwardly), the infinite activity case is treated by suitable approximations.

4.1 The case of finite activity

Definition 4.1. A generator function f satisfies condition (\mathbf{B}_γ) , if it is Lipschitz continuous in (y, z) , locally Lipschitz continuous in u (in the sense that $u \mapsto f_t(y, z, -c \vee u \wedge c)$ is Lipschitz continuous for any $c \in (0, \infty)$), $f.(0, 0, 0)$ is bounded, and f satisfies (\mathbf{A}_γ) .

The next statement readily leads to Theorem 4.3, for A in (2.6) with $\lambda(A) < \infty$.

Proposition 4.2. Let $\xi \in L^\infty(\mathcal{F}_T)$ and f satisfies (\mathbf{B}_γ) . Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) . Moreover for all $t \in [0, T]$, $|Y_t|$ is bounded by $\exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f.(0, 0, 0)|_\infty)$.

Proof. Consider the Lipschitz generator $f_t^c(y, z, u) := f_t(y, z, (u \vee (-c)) \wedge c)$ with $c > 0$ and Lipschitz constant K_{f^c} . By classical fixed point arguments and a-priori estimates (e.g. [Bec06, Prop.3.2, 3.3]), there exists a unique solution (Y^c, Z^c, U^c) in $\mathcal{S}^2 \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f^c) and it satisfies

$$|Y_t^c| \leq C\mathbb{E}\left(|\xi|^2 + \int_t^T |f_s^c(0, 0, 0)|^2 ds \mid \mathcal{F}_t\right) \leq C(|\xi|_\infty^2 + T|f.(0, 0, 0)|_\infty^2) < \infty,$$

for some constant $C = C(T, K_{f^c})$. Now Proposition 3.10 implies that $|Y_t^c|$ is dominated by $\exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f.(0, 0, 0)|_\infty)$ for all $c > 0$. Choosing $c \geq 2 \exp(K_f^{y,z}T)(|\xi|_\infty + T|f.(0, 0, 0)|_\infty)$ we get that (Y^c, Z^c, U^c) with $Y^c \in \mathcal{S}^\infty$ solves the BSDE (ξ, f) since U^c is bounded by c . Uniqueness follows by comparison. \square

This leads to a preliminary result on bounded solutions if jumps are of finite activity.

Theorem 4.3. Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f satisfy $(\mathbf{A}_{\text{fin}})$ (recall Definition 3.6) with $f.(0, 0, 0)$ bounded. Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) . Moreover for all $t \in [0, T]$, $|Y_t|$ is bounded by $\exp(K_f^{y,z}(T-t))(|\xi|_\infty + (T-t)|f.(0, 0, 0)|_\infty)$.

Proof. Noting that local Lipschitz continuity in u follows from the absolute continuity of g in u with locally bounded density function, the claim follows from Theorem 3.11 and Proposition 4.2. \square

Corollary 4.4. Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f be a generator satisfying $(\mathbf{A}_{\text{fin}})$, with $g_t(y, z, 0, e) \equiv 0$ and $|\hat{f}_t(y, z)| \leq K_1 + K_2|y|$ for some $K_1, K_2 \geq 0$. Set

$$b(t) = \begin{cases} (|\xi|_\infty + \frac{K_1}{K_2}) \exp(K_2(T-t)) - \frac{K_1}{K_2}, & K_2 \neq 0 \\ |\xi|_\infty + K_1(T-t), & K_2 = 0. \end{cases}$$

Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) and moreover $|Y_t| \leq b_t$ for $t \in [0, T]$. Finally $\int Z dB$ and $U * \tilde{\mu}$ are BMO(\mathbb{P})-martingales.

Proof. By Lemma 3.12 and Theorem 4.3, there is a unique solution (Y, Z, U) in the space $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, \tilde{f}) . By Proposition 3.13, it also solves the BSDE (ξ, f) and $-b(t) \leq Y_t \leq b(t)$, $\forall t \in [0, T]$. Uniqueness follows from the fact that one can apply the comparison Theorem 3.9 for generators satisfying $(\mathbf{A}_{\text{fin}})$. The BMO property follows from Lemma 2.3. \square

Remark 4.5. 1. While the statement of Corollary 4.4 is similar to Thm. 3.5 in [Bec06], its proof is different in that it relies on previous comparison results for JBSDEs instead of stopping arguments; Further comments about differences in technical conditions can be found in [Ken15, Rem.1.20].

2. The stochastic integrals of the BSDE solution are BMO-martingales under the assumptions for Lemma 2.3. In particular, this holds for instance under the conditions for Thm. 3.6 in [Bec06] where $\lambda(A) < \infty$, \hat{f} is linearly bounded in y and g is locally Lipschitz in u with $g_t(y, z, 0, e) \equiv 0$.

Corollary 4.6. *Let $\xi \in L^\infty(\mathcal{F}_T)$ with $\xi \geq C$ for some constant $C > 0$, $K \geq 0$ and set $a(t) := C \exp(-K(T-t))$ and $b(t) = |\xi|_\infty \exp(K(T-t))$, $\forall t \in [0, T]$. Assume f satisfies $(\mathbf{A}_{\text{fin}})$ for $c \leq y, y+u \leq d$ for all $c, d \in \mathbb{R}$ with $0 < c < d$, and that $|\hat{f}_t(y, z)| \leq K|y|$ and $g_t(y, z, 0, e) = 0$. Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) with $Y \geq \epsilon$ for some $\epsilon > 0$. Moreover, it holds $a(t) \leq Y_t \leq b(t)$ and $\int Z dB$ and $U * \tilde{\mu}$ are BMO(\mathbb{P})-martingales.*

Proof. This can be shown with a similar argument for the uniqueness as above: Let (Y', Z', U') be another solution to the BSDE (ξ, f) with $Y' \geq \epsilon$ for some $\epsilon > 0$. Then f satisfies $(\mathbf{A}_{\text{fin}})$ for $a(t) \wedge \epsilon \leq y, y+u \leq b(t) \vee |Y'|_\infty$; hence the solutions coincide by comparison. \square

Example 4.7. *As a special case of Corollary 4.6 to be applied in Section 5.2, setting $K := (\gamma|\varphi|_\infty^2)/(2(1-\gamma)^2)$ for some $\gamma \in (0, 1)$ and some predictable and bounded process φ we define*

$$\begin{aligned} f_t(y, z, u) &:= \hat{f}_t(y, z) + \int_E g_t(y, u, e) \zeta(t, e) \lambda(de) \\ &:= \frac{\gamma}{2(1-\gamma)^2} |\varphi_t|^2 y + \int_E \left(\frac{1}{1-\gamma} ((u(e) + y)^{1-\gamma} y^\gamma - y) - u(e) \right) \zeta(t, e) \lambda(de). \end{aligned}$$

From $\frac{\partial g}{\partial y}(t, y, u, e) = \left(\frac{u+y}{y}\right)^{1-\gamma} + \frac{\gamma}{1-\gamma} \left(\frac{u+y}{y}\right)^{-\gamma} - \frac{1}{1-\gamma}$, we see that f is Lipschitz in y within the truncation bounds. Moreover, g is continuously differentiable with bounded derivatives and we have $\frac{\partial g}{\partial u}(t, y, u, e) = \left(\frac{u+y}{y}\right)^{-\gamma} - 1 > -1$, for $c \leq y, y+u \leq d$.

4.2 The case of infinite activity

For linear generators of the form $f_t(y, z, u) := \alpha_t^0 + \alpha_t y + \beta_t z + \int_E \gamma_t(e) u(e) \zeta(t, e) \lambda(de)$, with predictable coefficients α^0, α, β and γ , JBSDE solutions can be represented by an adjoint process. In our context of bounded solutions, one needs rather weak conditions on the adjoint process. This will be used later on in Section 5. The idea of proof is standard, cf. [Ken15, Lem.1.23] for details.

Lemma 4.8. *Let f be a linear generator of the form above and $\xi \in L^\infty(\mathcal{F}_T)$.*

1. *Assume that $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ solves the BSDE (ξ, f) . Suppose that the adjoint process $(\Gamma_s^t)_{s \in [t, T]} := (\exp(\int_t^s \alpha_u du) \mathcal{E}(\int \beta dB + \gamma * \tilde{\mu})_t^s)_{s \in [t, T]}$ is in \mathcal{S}^1 for any $t \leq T$ and α^0 is bounded. Then Y is represented as $Y_t = \mathbb{E}[\Gamma_T^t \xi + \int_t^T \Gamma_s^t \alpha_s^0 ds | \mathcal{F}_t]$.*
2. *Let α^0, α, β and $\tilde{\gamma}_t := \int_E |\gamma_t(e)|^2 \zeta(t, e) \lambda(de)$, $t \in [0, T]$, be bounded and $\gamma \geq -1$. Then there is a unique solution in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) and Part 1. applies.*

Our aim is to prove existence and uniqueness beyond Theorem 4.3, for possibly infinite activity of jumps, i.e. $\lambda(A)$ may be infinite in (2.6). To show Theorems 4.11 and 4.13, we use a monotone stability approach of [Kob00]: By approximating a generator f of the form (2.7) (with A such that $\lambda(A) = \infty$) by a sequence $(f^n)_{n \in \mathbb{N}}$ of the form (2.7) (with A_n such that $\lambda(A_n) < \infty$) for which solutions' existence is guaranteed, one gets that the limit of these solutions exist and it solves the BSDE with the original data. As in [Kob00], the monotone approximation approach is not easy in execution, a main problem usually being to prove strong convergence of the stochastic integral parts for the BSDE. By Proposition 4.9 convergence works for small terminal condition ξ . That is why we can not apply this Proposition directly to data $(\xi, f^n)_{n \in \mathbb{N}}$. Instead we sum (converging) solutions for small $1/N$ -fractions of the desired terminal condition. This is inspired by the iterative ansatz from [Mor10] for a particular generator. For our generator family, we adapt and elaborate proofs, using e.g. \mathcal{S}^1 -closeness for the strong approximation step. Compared to [Mor10], the analysis for our general family of JBSDEs offers further clarity and structural insight into what is really needed. It extends the scope of the BSDE stability approach [Kob00, Mor10], in particular with regards to non-Lipschitz dependencies in the jump-integrand, while the proof also appears more concise, showing comparable ease for the (usually laborious) strong approximation step in the setup under consideration. Differently to e.g. [EMN16, Mor10], no exponential transforms or convolutions are utilized here, as our generators are “quadratic” in U but not in Z .

Despite similarities at first sight, a closer look reveals that Theorem 4.11 is different both in method and in scope from [KTPZ15]. Indeed, [KTPZ15, Thm.5.4] prove existence for small terminal conditions by following the fixed point approach by [Tev08], whereas we show stability for small terminal conditions (Proposition 4.9) and apply this in a different pasting procedure, approximating not only terminal data but also generators. Here wellposedness of the approximating JBSDEs is obtained directly from classical theory by using comparison and estimates from Section 3, which are of interest in themselves and enable us to argue within uniform a-priori bounds for the approximating sequence. Concrete examples in Section 5 further demonstrate that not just the method of proof but also the scope of our results is different from [KTPZ15, LS14].

In more detail, the task for Theorem 4.11 is to construct generators $(f^{k,n})_{1 \leq k \leq N, n \in \mathbb{N}}$ and solutions $(Y^{k,n}, Z^{k,n}, U^{k,n})$ to the BSDEs with data $(\xi/N, f^{k,n})$ for N large enough such that $(Y^{k,n}, Z^{k,n}, U^{k,n})$ converges if $n \rightarrow \infty$ and $(Y^n, Z^n, U^n) := \sum_{k=1}^N (Y^{k,n}, Z^{k,n}, U^{k,n})$ solves the BSDE (ξ, f^n) . In this case (Y^n, Z^n, U^n) converges and its limit is a solution candidate for the BSDE (ξ, f) . For this program, we next show a stability result for JBSDE.

Proposition 4.9. *Let $(\xi^n) \subset L^\infty(\mathcal{F}_T)$ with $\xi^n \rightarrow \xi$ in $L^2(\mathcal{F}_T)$ and $(f^n)_{n \in \mathbb{N}}$ be a sequence of generators with $f^n(0, 0, 0) = 0$, $\forall n$, having property (B_{γ^n}) such that $K_f^{y,z} := \sup_{n \in \mathbb{N}} K_{f^n}^{y,z} < \infty$. Denote by $(Y^n, Z^n, U^n) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ the solution to the BSDE (ξ, f^n) with Y^n bounded by $|\xi|_\infty \exp(K_f^{y,z} T)$ and set $\tilde{c} := |\xi|_\infty \exp(K_f^{y,z} T)$. Assume that Y^n converges pointwise, $(Z^n, U^n) \rightarrow (Z, U)$ converges weakly in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ and $|f_t^n(0, 0, u)| \leq \hat{K}|u|_t^2 + \hat{L}_t$ for all n and u with $|u| \leq 2\tilde{c}$, $\hat{K} \in \mathbb{R}_+$ and $\hat{L} \in L^1(\mathbb{P} \otimes dt)$. Then (Z^n, U^n) converges to (Z, U) strongly in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$, if $|\xi|_\infty \equiv \tilde{c} \exp(-K_f^{y,z} T) \leq \exp(-K_f^{y,z} T)/(80 \max\{K_f^{y,z}, \hat{K}\})$.*

Proof. We note that (Y^n, Z^n, U^n) is uniquely defined by Proposition 4.2. To prove strong convergence of $(Z^n)_{n \in \mathbb{N}}$ and $(U^n)_{n \in \mathbb{N}}$ we consider $\delta Y := Y^n - Y^m$, $\delta Z := Z^n - Z^m$, $\delta U := U^n - U^m$ and apply Itô's formula for general semimartingales to $(\delta Y)^2$ to obtain

$$\begin{aligned} (\delta Y_0)^2 &= (\delta Y_T)^2 + \int_0^T 2\delta Y_{s-} (f_s^n(Y_{s-}^n, Z_s^n, U_s^n) - f_s^m(Y_{s-}^m, Z_s^m, U_s^m)) ds \\ &\quad - \int_0^T \|\delta Z_s\|^2 ds - 2 \int_0^T \delta Y_{s-} \delta Z_s dB_s - \int_0^T \int_E (\delta Y_{s-} + \delta U_s(e))^2 - (\delta Y_{s-})^2 \tilde{\mu}(ds, de) \\ &\quad - \int_0^T \int_E (\delta Y_{s-} + \delta U_s(e))^2 - (\delta Y_{s-})^2 - 2\delta Y_{s-} \delta U_s(e) \nu(ds, de). \end{aligned}$$

Noting that the stochastic integrals are martingales one concludes that

$$\begin{aligned} &\mathbb{E} \left(\int_0^T 2\delta Y_{s-} (f_s^n(Y_{s-}^n, Z_s^n, U_s^n) - f_s^m(Y_{s-}^m, Z_s^m, U_s^m)) ds \right) \\ &= \mathbb{E} \left(\int_0^T \int_E \delta U_s(e)^2 \nu(ds, de) \right) + \mathbb{E} \left(\int_0^T \|\delta Z_s\|^2 ds \right) - \mathbb{E}((\delta Y_T)^2) + \mathbb{E}((\delta Y_0)^2). \end{aligned} \quad (4.1)$$

Using the inequalities $a \leq a^2 + 1/4$, $(a+b)^2 \leq 2(a^2 + b^2)$, $(a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$, the Lipschitz property of f^n in y and z and the estimate for $f_t^n(0, 0, u)$, we have

$$\begin{aligned} &|f_s^n(Y_{s-}^n, Z_s^n, U_s^n) - f_s^m(Y_{s-}^m, Z_s^m, U_s^m)| \\ &\leq K_{f^n}^{y,z} (|Y_{s-}^n| + \|Z_s^n\|) + K_{f^m}^{y,z} (|Y_{s-}^m| + \|Z_s^m\|) + \hat{K}|U_s^n|_s^2 + \hat{L}_s + \hat{K}|U_s^m|_s^2 + \hat{L}_s \\ &\leq K_1 + 2\hat{L}_s + K_2(\|\delta Z_s\|^2 + \|Z_s^n - Z_s^m\|^2 + \|Z_s^m\|^2 + |\delta U_s|_s^2 + |U_s^n - U_s^m|_s^2 + |U_s^m|_s^2), \end{aligned} \quad (4.2)$$

where $K_1 := K_f^{y,z}(2\tilde{c} + 1/2) \in \mathbb{R}$, $K_2 := 5 \max\{K_f^{y,z}, \hat{K}\}$ and $|\cdot|_t$ is defined in (2.5). Combing inequalities (4.1) and (4.2) yields

$$\begin{aligned} \mathbb{E} \left(\int_0^T \|\delta Z_s\|^2 + |\delta U_s|_s^2 ds \right) &\leq 2\mathbb{E} \left(\int_0^T |\delta Y_{s-}| (K_1 + 2\hat{L}_s + K_2(\|\delta Z_s\|^2 + \|Z_s^n - Z_s^m\|^2 + \|Z_s^m\|^2 \right. \\ &\quad \left. + |\delta U_s|_s^2 + |U_s^n - U_s^m|_s^2 + |U_s^m|_s^2)) ds \right) + \mathbb{E}((\xi^n - \xi^m)^2). \end{aligned}$$

Let us recall that the predictable projection of Y , denoted by Y^p , is defined as the unique predictable process X such that $X_\tau = \mathbb{E}(Y_\tau | \mathcal{F}_{\tau-})$ on $\{\tau < \infty\}$ for all predictable times τ . For Y^n it holds $(Y^n)^p = Y_-^n$. This follows from [JS03, Prop.I.2.35.] using that Y^n is càdlàg, adapted and quasi-left-continuous, as $\Delta Y_\tau = \Delta U * \tilde{\mu}_\tau = 0$ on $\{\tau < \infty\}$ holds for all predictable times τ thanks to the absolute continuity of the compensator ν . Noting that $1 - 2K_2|\delta Y_{s-}| \geq 1 - 4K_2\tilde{c} \geq 3/4$ and setting $Y := \lim_{n \rightarrow \infty} Y^n$ we deduce by the weak convergence of $(Z^n)_{n \in \mathbb{N}}$ and $(U^n)_{n \in \mathbb{N}}$, $Y_-^n = (Y^n)^p \uparrow (Y)^p$ as $n \rightarrow \infty$ and by Lebesgue's dominated convergence theorem

$$\begin{aligned}
& \frac{3}{4} \mathbb{E} \left(\int_0^T \|Z_s^n - Z_s\|^2 + |U_s^n - U_s|_s^2 ds \right) \\
& \leq \frac{3}{4} \liminf_{m \rightarrow \infty} \mathbb{E} \left(\int_0^T \|Z_s^n - Z_s^m\|^2 + |U_s^n - U_s^m|_s^2 ds \right) \\
& \leq \liminf_{m \rightarrow \infty} 2 \mathbb{E} \left(\int_0^T |\delta Y_{s-}| (K_1 + 2\widehat{L}_s + K_2(\|Z_s^n - Z_s\|^2 + \|Z_s\|^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2)) ds \right) \\
& \quad + \mathbb{E}((\xi^m - \xi^n)^2) \\
& = 2 \mathbb{E} \left(\int_0^T |Y_{s-}^n - (Y_s)^p| (K_1 + 2\widehat{L}_s + K_2(\|Z_s^n - Z_s\|^2 + \|Z_s\|^2 + |U_s^n - U_s|_s^2 + |U_s|_s^2)) ds \right) \\
& \quad + \mathbb{E}((\xi - \xi^n)^2).
\end{aligned}$$

Noting $3/4 - 2K_2|Y_{s-}^n - (Y_s)^p| \geq 3/4 - 4K_2\tilde{c} \geq 1/2$, one obtains with dominated convergence

$$\begin{aligned}
& \frac{1}{2} \limsup_{n \rightarrow \infty} \mathbb{E} \left(\int_0^T \|Z_s^n - Z_s\|^2 + |U_s^n - U_s|_s^2 ds \right) \\
& \leq \limsup_{n \rightarrow \infty} 2 \mathbb{E} \left(\int_0^T |Y_{s-}^n - (Y_s)^p| (K_1 + 2\widehat{L}_s + \|Z_s\|^2 + |U_s|_s^2) ds \right) + \mathbb{E}((\xi^n - \xi)^2) = 0.
\end{aligned}$$

□

We will need the following result which is a slight variation of [Kob00, Lem.2.5].

Lemma 4.10. *Let $(Z^n)_{n \in \mathbb{N}}$ be convergent in $\mathcal{L}^2(B)$ and $(U^n)_{n \in \mathbb{N}}$ convergent in $\mathcal{L}^2(\tilde{\mu})$. Then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $\sup_{n_k} \|Z^{n_k}\| \in L^2(\mathbb{P} \otimes dt)$ and $\sup_{n_k} |U_t^{n_k}|_t \in L^2(\mathbb{P} \otimes dt)$.*

Proof. The result for $(Z^n)_{n \in \mathbb{N}}$ is from [Kob00] and the argument for $(U^n)_{n \in \mathbb{N}}$ is analogous. □

Theorem 4.11 (Infinite activity). *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let $(f^n)_n$ be a sequence of drivers satisfying condition (B_{γ^n}) with $K_f^{y,z} := \sup_{n \in \mathbb{N}} K_{f^n}^{y,z} < \infty$. Assume that*

1. *there is $(\widehat{Y}, \widehat{Z}, \widehat{U})$ in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ with \widehat{U} bounded and $f_t^n(\widehat{Y}_{t-}, \widehat{Z}_t, \widehat{U}_t) \equiv 0$ for all n ,*
2. *for all $u \in L^0(\mathcal{B}(E), \lambda)$ with $|u| \leq |\widehat{U}|_\infty + 2|\xi|_\infty \exp(K_f^{y,z}T)$ there exists $\widehat{K} \in \mathbb{R}_+$ and a process $\widehat{L} \in L^1(\mathbb{P} \otimes dt)$ such that $|f_t^n(0, 0, u)| \leq \widehat{K}|u|_t^2 + \widehat{L}_t$ for each $n \in \mathbb{N}$,*
3. *the sequence $(f^n)_{n \in \mathbb{N}}$ converges pointwise and monotonically to a generator f ,*
4. *there is a BMO(\mathbb{P})-martingale M such that for all truncated generators $f_t^{n,\hat{c}}(y, z, u) := f_t^n((y \vee (-\hat{c})) \wedge \hat{c}, z, (u \vee (-2\hat{c})) \wedge (2\hat{c}))$ with $\hat{c} := |\widehat{Y}|_\infty + (|\widehat{U}|_\infty/2) + \exp(K_f^{y,z}T)|\xi|_\infty$ holds $\int_t^T f_s^{n,\hat{c}}(Y_{s-}, Z_s, U_s) ds \leq \langle M \rangle_T - \langle M \rangle_t$ or $-\int_t^T f_s^{n,\hat{c}}(Y_{s-}, Z_s, U_s) ds \leq \langle M \rangle_T - \langle M \rangle_t$ for all $n \in \mathbb{N}$, $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$, and*
5. *for all $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ and $(U^n)_{n \in \mathbb{N}} \in \mathcal{L}^2(\tilde{\mu})$ with $U^n \rightarrow U$ in $L^2(\tilde{\mu})$ it holds $f^n(Y_-, Z, U^n) \rightarrow f(Y_-, Z, U)$ in $L^1(\mathbb{P} \otimes dt)$.*

Then

- i) *there exists a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ for the BSDE (ξ, f) , with $\int Z dB$ and $U * \tilde{\mu}$ being BMO(\mathbb{P})-martingales, and*

ii) this solution is unique if additionally f satisfies (\mathbf{A}'_γ) .

Proof. Let us first outline the overall program of the proof. We want to construct generators $(f^{k,n})_{1 \leq k \leq N, n \in \mathbb{N}}$ and solutions $(Y^{k,n}, Z^{k,n}, U^{k,n})$ to the BSDEs $(\xi/N, f^{k,n})$ for N sufficiently large (to employ Proposition 4.9 and get that $((Y^{k,n}, Z^{k,n}, U^{k,n}))_{n \in \mathbb{N}}$ converges and $(Y^n, Z^n, U^n) := \sum_{k=1}^N (Y^{k,n}, Z^{k,n}, U^{k,n})$ solves the BSDE (ξ, f^n)). We show that if for some $k < N$ and all $1 \leq l \leq k$ and $n \in \mathbb{N}$ we have already constructed generators $(f^{l,n})_{1 \leq l \leq k, n \in \mathbb{N}}$ such that there exists solutions $((Y^{l,n}, Z^{l,n}, U^{l,n}))_{n \in \mathbb{N}}$ to the BSDEs $(\xi/N, f^{l,n})$ converging for $n \rightarrow \infty$, with $|Y^{l,n}|_\infty \leq \exp(K_f^{y,z} T) |\xi|_\infty / N =: \tilde{c}$, then for $\bar{Y}^{k,n} := \hat{Y} + \sum_{l=1}^k Y^{l,n}$ with $\bar{Z}^{k,n}$ and $\bar{U}^{k,n}$ defined analogously and

$$f_t^{k+1,n}(y, z, u) := f_t^n(y + \bar{Y}_{t-}^{k,n}, z + \bar{Z}_{t-}^{k,n}, u + \bar{U}_{t-}^{k,n}) - f_t^n(\bar{Y}_{t-}^{k,n}, \bar{Z}_{t-}^{k,n}, \bar{U}_{t-}^{k,n}) \quad (4.3)$$

there are solutions $(Y^{k+1,n}, Z^{k+1,n}, U^{k+1,n}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDEs $(\xi/N, f^{k+1,n})$, converging (in n) and satisfying $|Y^{k+1,n}|_\infty \leq \tilde{c}$. Starting initially with the triple $(Y^{0,n}, Z^{0,n}, U^{0,n})$ defined by $(Y^{0,n}, Z^{0,n}, U^{0,n}) := (\hat{Y}, \hat{Z}, \hat{U})$, formula (4.3) gives an inductive construction of the generators $f^{k,n}$ and triples $(Y^{k,n}, Z^{k,n}, U^{k,n}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ solving the BSDE $(\xi/N, f^{k,n})$ and converging for $n \rightarrow \infty$ with $|Y^{k,n}|_\infty \leq \tilde{c}$ for each $n \in N$ and $1 \leq k \leq N$.

Note that $f^{k+1,n}$ is Lipschitz continuous in y and z with Lipschitz constant $K_{f^n}^{y,z}$, locally Lipschitz in u and satisfies condition $(A_{\gamma^{k+1,n}})$ with

$$\gamma_s^{k+1,n}(y, z, u, u', e) := \gamma_s^n(y + \bar{Y}_{s-}^{k,n}, z + \bar{Z}_{s-}^{k,n}, u + \bar{U}_{s-}^{k,n}(e), u' + \bar{U}_{s-}^{k,n}(e), e)$$

and $f_t^{k+1,n}(0, 0, 0) \equiv 0$. Hence by the existence and uniqueness result for the finite activity case (see Proposition 4.2), there exists a unique solution $(Y^{k+1,n}, Z^{k+1,n}, U^{k+1,n})$ to the BSDE $(\xi/N, f^{k+1,n})$ such that $Y^{k+1,n}$ is bounded by \tilde{c} .

To apply Proposition 4.9, we have to check that the sequence $(Y^{k+1,n})_{n \in \mathbb{N}}$ converges pointwise, that $(Z^{k+1,n}, U^{k+1,n})_{n \in \mathbb{N}}$ converges weakly in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ and that $f^{k+1,n}(0, 0, u)$ can be locally bounded by an affine function in $|u|^2$. Having telescoping sums in (4.3) implies that $(\bar{Y}^{l,n}, \bar{Z}^{l,n}, \bar{U}^{l,n})$ solves the BSDE $(\hat{Y}_T + l\xi/N, f^n)$. By the comparison result of Proposition 3.1, the sequences $(\bar{Y}^{k,n})_{n \in \mathbb{N}}$ and $(\bar{Y}^{k+1,n})_{n \in \mathbb{N}}$ are monotonic (and bounded) in n so that finite limits $\lim_{n \rightarrow \infty} Y^{k+1,n} = \lim_{n \rightarrow \infty} \bar{Y}^{k+1,n} - \lim_{n \rightarrow \infty} \bar{Y}^{k,n}$ exists, $\mathbb{P} \otimes dt$ -a.e.. By Lemma 2.3, the sequences $(\bar{Z}^{k,n}, \bar{U}^{k,n})_{n \in \mathbb{N}}$ and $(\bar{Z}^{k+1,n}, \bar{U}^{k+1,n})_{n \in \mathbb{N}}$ are bounded in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$; hence $(Z^{k+1,n}, U^{k+1,n})$ is weakly convergent in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ along a subsequence which we still index by n for simplicity. Due to the Lipschitz continuity of f^n and Assumption 2., we get for all $|u| \leq 2\tilde{c}$ that

$$\begin{aligned} |f_t^{k+1,n}(0, 0, u)| &\leq |f_t^n(\bar{Y}_{t-}^{k,n}, \bar{Z}_{t-}^{k,n}, u + \bar{U}_{t-}^{k,n}) - f_t^n(\bar{Y}_{t-}^{k,n}, \bar{Z}_{t-}^{k,n}, \bar{U}_{t-}^{k,n})| \\ &\leq 2K_{f^n}^{y,z} (|\bar{Y}_{t-}^{k,n}| + \|\bar{Z}_{t-}^{k,n}\|) + \hat{K}(|u + \bar{U}_{t-}^{k,n}|_t^2 + |\bar{U}_{t-}^{k,n}|_t^2) + 2\hat{L}_t \\ &\leq 2\hat{K}|u|_t^2 + \tilde{L}_t, \end{aligned}$$

where $\tilde{L}_t = 2K_f^{y,z}(\hat{c} + \sup_{n \in \mathbb{N}} \|\bar{Z}_t^{k,n}\|^2 + 1/4) + 3\hat{K} \sup_{n \in \mathbb{N}} |\bar{U}_t^{k,n}|_t^2 + 2\hat{L}_t$. Here we used that by induction hypothesis $(\bar{Z}^{k,n}, \bar{U}^{k,n})_n$ is convergent so that $\sup_{n \in \mathbb{N}} (\|\bar{Z}_t^{k,n}\|^2 + |\bar{U}_t^{k,n}|_t^2)$ is $\mathbb{P} \otimes dt$ -integrable by Lemma 4.10 along a subsequence which again for simplicity we still index by n . This implies that $\tilde{L} \in L^1(\mathbb{P} \otimes dt)$, and therefore by Proposition 4.9, the sequence $(Z^n, U^n) := (\bar{Z}^{N,n}, \bar{U}^{N,n})$ converges in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to some (Z, U) in $\mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ while $(Y^n) := (\bar{Y}^{N,n})$ converges to some Y . Hence, $f^n(Y_-^n, Z^n, U^n) - f^n(Y_-, Z, U^n)$ converges to 0 in $L^1(\mathbb{P} \otimes dt)$ and by Assumption 5. we have $f^n(Y_-^n, Z^n, U^n) \rightarrow f(Y_-, Z, U)$ in $L^1(\mathbb{P} \otimes dt)$. The stochastic integrals $(Z^n - Z^m) \bullet B$ and $(U^n - U^m) \bullet \tilde{\mu}$ belong to $\mathcal{S}^2 \subset \mathcal{S}^1$ by Doob's inequality, with \mathcal{S}^1 -norms being bounded by a multiple of $\|Z^n - Z^m\|_{\mathcal{L}^2(B)}$ and $\|U^n - U^m\|_{\mathcal{L}^2(\tilde{\mu})}$ respectively. Since $|Y^n - Y^m|_{\mathcal{S}^1}$ is dominated by

$$\|f^n(Y_-^n, Z^n, U^n) - f^m(Y_-^m, Z^m, U^m)\|_{L^1(\mathbb{P} \otimes dt)} + C(\|Z^n - Z^m\|_{\mathcal{L}^2(B)} + \|U^n - U^m\|_{\mathcal{L}^2(\tilde{\mu})})$$

for some constant $C > 0$ with the bound tending to 0 as $n, m \rightarrow \infty$, we can take Y in \mathcal{S}^1 due to completeness of \mathcal{S}^1 ; see [DM82, VII. 3.64].

Finally, (Y, Z, U) solves the BSDE (ξ, f) since the approximating solutions $(Y^n, Z^n, U^n)_{n \in \mathbb{N}}$ of the BSDE $(\xi, f^n)_{n \in \mathbb{N}}$ converge to some $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ and $f^n(Y_-, Z_-, U_-)$ tends to $f(Y_-, Z_-, U_-)$ in $L^1(\mathbb{P} \otimes dt)$. Hence, we have $\int_0^t f_s^n(Y_{s-}, Z_{s-}, U_{s-}) ds \rightarrow \int_0^t f_s(Y_{s-}, Z_{s-}, U_{s-}) ds$, $\int_0^t Z_s^n dB_s \rightarrow \int_0^t Z_s dB_s$ and $U^n * \tilde{\mu}_t \rightarrow U * \tilde{\mu}_t$ \mathbb{P} -a.s. (along a subsequence) for all $0 \leq t \leq T$. \square

The next corollary to Theorem 4.11 provides conditions under which the Z -part of the JBSDE solution vanishes. Such is useful for applications in a pure-jump context (see e.g. Section 5.1.2 or [CFJ16]) with weak PRP by $\tilde{\mu}$ alone (cf. Example 2.1-1., 3., 4.), without a Brownian motion. Clearly an independent Brownian motion can always be added by enlarging the probability space, but it is then natural to ask for a JBSDE solution with trivial Z -part, adapted to the original filtration. Instead of re-doing the entire argument leading to Theorem 4.11 but now for JBSDEs solely driven by a random measure $\tilde{\mu}$ with generators without a z -argument, the next result gives a direct argument to this end. An example where the corollary is applied is given in Section 5.1.2.

Corollary 4.12. *Let $\mu = \mu^X$ be the random measure associated to a pure-jump process X , such that the compensated random measure $\tilde{\mu}$ alone has the weak PRP (see (2.2)) with respect to the usual filtration \mathbb{F}^X of X . Let B be a d -dimensional Brownian motion independent of X . With respect to $\mathbb{F} := \mathbb{F}^{B, X}$, let $f, (f^n)_n, \xi$ satisfy the assumptions of Theorem 4.11 with $\tilde{Z} = 0$ and f satisfying (A'_γ) . Let ξ be in $L^\infty(\mathcal{F}_T^X)$ and f, f^n be $\mathcal{P}(\mathbb{F}^X) \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{B}(E)))$ -measurable. Then the JBSDE (ξ, f) admits a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$, and we have that Y is \mathbb{F}^X -adapted, $Z = 0$, and U can be taken as measurable with respect to $\tilde{\mathcal{P}}(\mathbb{F}^X)$.*

Proof. Let B' be a (1-dimensional) Brownian motion independent of (B, X) . Then $\bar{B} := (B, B')$ is a $(d+1)$ -dimensional Brownian motion independent of X . Let $\mathbb{F}' := \mathbb{F}^{B', X}$ and $\bar{\mathbb{F}} := \mathbb{F}^{\bar{B}, X}$ denote the usual filtrations of (B', X) and (\bar{B}, X) . As in Example 2.1.3., $(B, \tilde{\mu})$, $(B', \tilde{\mu})$ and $(\bar{B}, \tilde{\mu})$ each admits the weak PRP w.r.t. \mathbb{F}, \mathbb{F}' and $\bar{\mathbb{F}}$ respectively. Now consider the generator function \tilde{f} that does not depend on z and is defined by $\tilde{f}_t(y, u) := f_t(y, 0, u)$. Because $\tilde{Z} = 0$, the conditions for Theorem 4.11 are met by $\tilde{f}^n := f^n(\cdot, 0, \cdot)$. In addition, \tilde{f} satisfies condition (A'_γ) since f does. Since ξ is \mathcal{F}_T^X -measurable and \tilde{f} is $\mathcal{P}(\mathbb{F}^X) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(L^0(\mathcal{B}(E)))$ -measurable, then by Theorem 4.11 the JBSDE (ξ, \tilde{f}) simultaneously admits unique solutions (Y, Z, U) , (Y', Z', U') and $(\bar{Y}, \bar{Z}, \bar{U})$ in the respective $\mathcal{S}^\infty \times \mathcal{L}^2(\cdot) \times \mathcal{L}^2(\tilde{\mu})$ -spaces for each of the filtrations \mathbb{F}, \mathbb{F}' and $\bar{\mathbb{F}}$. Noting that both \mathbb{F} and \mathbb{F}' are sub-filtrations of $\bar{\mathbb{F}}$, we get by uniqueness of $(\bar{Y}, \bar{Z}, \bar{U})$ that $Z \bullet B = Z' \bullet B' = \bar{Z} \bullet \bar{B}$ and that Y is \mathbb{F}^X -adapted. The former implies $Z = Z' = 0$ by the strong orthogonality of B and B' . The claim follows, by noting that the JBSDE gives the (unique) canonical decomposition of the special semimartingale Y and using weak predictable martingale representation in \mathbb{F}^X . \square

A natural ansatz to approximate an f of the form (2.7) with $\lambda(A) = \infty$ is by taking

$$f_t^n(y, z, u) := \hat{f}_t(y, z) + \int_{A_n} g_t(u(e), e) \zeta(t, e) \lambda(de), \quad (4.4)$$

for an increasing sequence $(A_n)_{n \in \mathbb{N}} \uparrow A$ of measurable sets with $\lambda(A_n) < \infty$ (as λ is σ -finite).

Theorem 4.13. *Let generator f be of the form (2.7) and $\xi \in L^\infty(\mathcal{F}_T)$. Let \hat{f} be Lipschitz continuous with respect to (y, z) uniformly in (t, ω, u) , and let $g(t, u, e)$ be absolutely continuous in u for all (ω, t, e) , with density function $g'(t, x, e)$ being strictly greater than -1 and locally bounded (in u) from above uniformly in (ω, t, y, z, e) . Assume that*

1. *there exists $(\hat{Y}, \hat{Z}, \hat{U}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ with $|\hat{U}|_\infty < \infty$, $\hat{f}_t(\hat{Y}_t, \hat{Z}_t) \equiv 0$ and $g_t(\hat{U}_t(e), e) \equiv 0$,*
2. *the function g is locally bounded in $|u|^2$ uniformly in (ω, t, e) , i.e. locally (in u) there exists a K such that $|g_t(u, e)| \leq K|u|^2$, and*
3. *there exists $D : \mathbb{R} \mapsto \mathbb{R}$ continuous such that $g \geq 0$ and either $\hat{f}_t(y, z) \geq D(y)$ for $|y| \leq \hat{c} := |\hat{Y}|_\infty + (|\hat{U}|_\infty/2) + |\xi|_\infty \exp(K_f^{y,z} T)$, or $g \leq 0$ and $\hat{f}_t(y, z) \leq D(y)$ for $|y| \leq \hat{c}$.*

Then

- i) there exists a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ and for each solution triple the stochastic integrals $\int Z dB$ and $U * \tilde{\mu}$ are BMO-martingales, and
- ii) this solution is unique if moreover the function g satisfies $(\mathbf{A}_{\text{inf}})$.

Finally, the same statements hold if condition 1. is replaced by assuming that f is not depending on y , i.e. $f_t(y, z, u) = f_t(z, u)$, and that \hat{f} is bounded.

Proof. We check that the assumptions of Theorem 4.11 are satisfied. Clearly conditions 1. and 2. are sufficient for assumptions 1. and 2. in Theorem 4.11. For f^n given by (4.4), the sequence (f^n) is either monotone increasing or monotone decreasing, depending on the sign of g . For the next assumption 4., $f^{n, \hat{c}}$ is bounded from above (or resp. below) by $\sup_{|y| \leq \hat{c}} D(y)$ (respectively $\inf_{|y| \leq \hat{c}} D(y)$). To show that also condition 5. of Theorem 4.11 holds, we prove that $g_t(U_t^n(e), e) \mathbb{1}_{A_n}(e)$ converge to $g_t(U_t(e), e)$ in $L^1(\mathbb{P} \otimes \nu)$ as $n \rightarrow \infty$ for $U^n \rightarrow U$ in $\mathcal{L}^2(\tilde{\mu})$, recalling (2.1). We set $B_n := (g_t(U_t^n(e), e) - g_t(U_t(e), e)) \mathbb{1}_{A_n}(e)$ and $C_n := g_t(U_t(e), e) \mathbb{1}_{A_n^c}(e)$. Both sequences $(B_n)_{n \in \mathbb{N}}$ and $(C_n)_{n \in \mathbb{N}}$ converge to 0 $\mathbb{P} \otimes \nu$ -a.e. since $U^n \rightarrow U$ in $L^2(\mathbb{P} \otimes \nu)$, g is locally Lipschitz in u and $A_n^c \downarrow \emptyset$. Moreover, they are bounded by integrable random variables. In particular, B_n is bounded by $\hat{K}(\sup_{n \in \mathbb{N}} |U_t^n(e)|^2 + |U_t(e)|^2)$ for some $\hat{K} > 0$ which is integrable along a subsequence due to Lemma 4.10. Hence applying the dominated convergence theorem yields the desired result.

In the alternative case without the Assumption 1., existence is still guaranteed. Indeed, let $f_t(y, z, u) = f_t(z, u)$ and \hat{f} be bounded. Denoting $\tilde{f}_t(z, u) := f_t(z, u) - f_t(0, 0)$ and $\tilde{\xi} := \xi + \int_0^T f_t(0, 0) dt$, there exists a unique solution (\tilde{Y}, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE $(\tilde{\xi}, \tilde{f})$ with $\int Z dB$ and $U * \tilde{\mu}$ being BMO-martingales by the first version of this theorem and noting that $g_t(0, e) \equiv 0$ and $f_t(0, 0) = \hat{f}_t(0)$ is bounded. Taking $Y_t := \tilde{Y}_t - \int_0^t \tilde{f}_s(0, 0) ds$, we obtain that (Y, Z, U) solves the BSDE with the data (ξ, f) . If moreover the function f satisfies $(\mathbf{A}_{\text{inf}})$, then f satisfies (\mathbf{A}'_γ) (cf. Example 3.8.3.) and hence uniqueness follows from applicability of the comparison argument in Proposition 3.1. \square

Example 4.14. A function g is locally bounded in u^2 in the sense of condition 2. in Theorem 4.13 if, for instance, $u \mapsto g_t(u, e)$ is twice differentiable for any (ω, t, e) , with the second derivative in u being locally bounded uniformly in (ω, t, e) , and $g_t(0, e) \equiv g'_t(0, e) \equiv 0$ vanishing.

Remark 4.15. Related results on wellposedness and comparison of JBSDEs are also provided in [KTPZ15] (further applied in [KTPZ16]). Quite differently, many of their assumptions are stated in terms of (pointwise) inequalities of stochastic fields, to hold for ‘all (t, ω, y, z, u) ’ in, presumably, $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times L^2(\nu_t)$ with $d\nu_t = \zeta_t d\lambda$. Without going into technical details here, such conditions appear not easy to verify and more discussion of concrete examples would seem helpful to understand the scope. At least, such assumptions should require some subtle choice of versions for the postulated processes, and also for disintegration of the compensator cf. [JS03, Sect. II.1.9]. In contrast, conditions for our theorems are on functions g that have Euclidean arguments, appear easier to check and are notably different in scope, as illustrated by concrete examples in Section 5.

Remark 4.16. In an similar setup with random measures for jumps, [LS14] use convexity techniques to study JBSDEs, whose generators can be also quadratic (in z). Their JBSDE results require different assumptions on the generator, notably convexity in z and u [LS14, Thm.A28, Cor.A29, which are further used in the proofs for Prop.A30, Thms.4.3, 4.5]. Many interesting applications are convex in nature, but not all. We show in Section 5.1.2 by a concrete example how indeed non-convex JBSDE generators arise for solving a utility maximization problem in a financial market under non-convex trading constraints. Note that convexity is not required for our theorems on comparison, existence and uniqueness of JBSDE.

5 Examples: optimal control problems in finance

To demonstrate the scope and applicability of previous results on JBSDEs as well as notable differences to some related recent literature, we solve now four distinct application problems of optimal control in finance and discuss concrete examples. The two problems in Section 5.1 are concerned with exponential utility maximization, possibly with an additive liability, and illustrate

how the general theory from previous sections on a broader family of JBSDEs apply in this problem, which is closely related to the entropic risk measure and has been a popular motivation for (quadratic, non-Lipschitz) JBSDE theory, cf. e.g. [Bec06, BEK09, Mor09, Bec10, KTPZ15, LS14]. Let us note that [Mor10] provided a detailed analysis of existence for the specific (quadratic) JBSDE of this particular application, with results being more general in some interesting aspects (compact constraints, jump-diffusion stock price) but less so in others (multiple assets, time inhomogeneous μ , unbounded controls). [Bec06] studied a related problem for jumps of finite activity.

Section 5.2 shows how a change of coordinates can transform a JBSDE that arises from an optimal control problem, but appears to be out of scope at first, into a JBSDE for which the theory of Section 4 can be applied to derive optimal controls from existence results for BSDEs, like in [HIM05, Sek06, Bec06]. To our best knowledge, the considered power-utility problem with jumps and multiplicative liability is solved for the first time in this spirit. Based on control theoretic arguments, [Nut12] provides a general analysis of power utility maximization, including a characterization of the so-called opportunity process in terms of a semimartingale BSDE, whose existence is inferred from existence of the optimal value process, obtained by some other means. Such an approach typically requires convexity conditions, e.g. for convex duality methods, which can be useful, of course, but are not a necessity for BSDE theory in general (cf. e.g. [HIM05] for non-convex constraints) and appear in some approaches, e.g. [LS14, Appendix A]. Finally, Section 5.3 derives a rigorous solution to the no-good-deal valuation problem [CSR00, BS06] in incomplete financial markets, which is posed over a multiplicatively stable sub-family of arbitrage-free pricing measures. This complements the analysis in [BS06] by JBSDE results for a non-Markovian setting. Also in this problem, whose (non-linear) JBSDE generator is even Lipschitz, the slight generalization of Proposition 3.1 to the classical comparison theorem by [Roy06] is helpful; Indeed, the process γ in (5.17) is such that the martingale condition (3.1) for Proposition 3.1 can be readily verified, while the same is not clear for the condition (\mathbf{A}_γ) in [Roy06, Thm.2.3].

For Sections 5.1.1, 5.2 and 5.3, we will consider models for a financial market within the framework of Section 2, consisting of one savings account with zero interest rate (for simplicity) and k risky assets ($k \leq d$), whose discounted prices evolve according to the stochastic differential equation

$$dS_t = \text{diag}(S_t^i)_{1 \leq i \leq k} \sigma_t (\varphi_t dt + dB_t) =: \text{diag}(S_t) dR_t, \quad t \in [0, T], \quad (5.1)$$

with $S_0 \in (0, \infty)^k$, where the market price of risk φ is a predictable \mathbb{R}^d -valued process, with $\varphi_t \in \text{Im } \sigma_t^T = (\text{Ker } \sigma_t)^\perp$ for all $t \leq T$, and σ is a predictable $\mathbb{R}^{k \times d}$ -valued process such that σ is of full rank k (i.e. $\det(\sigma_t \sigma_t^T) \neq 0$ $\mathbb{P} \otimes dt$ -a.e.) and integrable w.r.t. $\widehat{B} := B + \int_0^\cdot \varphi_t dt$. We take the market price of risk φ to be bounded $\mathbb{P} \otimes dt$ -a.e.. The market is free of arbitrage in the sense that the set \mathcal{M}^e of equivalent local martingale measures for S is non-empty. In particular, \mathcal{M}^e contains the minimal martingale measure

$$d\widehat{\mathbb{P}} := \mathcal{E}(-\varphi \bullet B)_T d\mathbb{P} = \exp\left(-\varphi \bullet B_T - \frac{1}{2} \int_0^T |\varphi_t|^2 dt\right) d\mathbb{P}, \quad (5.2)$$

under which \widehat{B} is a Brownian motion and S is a local martingale by Girsanov's theorem. Clearly, the market (5.1) is incomplete in general (even if $k = d$ and σ is invertible when the random measure is not trivial, filtration then being non-Brownian), cf. Example 2.1.

5.1 Exponential utility maximization

For a market with stock prices as in (5.1), consider the expected utility maximization problem

$$v_t(x) = \text{ess sup}_{\theta \in \Theta} \mathbb{E}(u(X_T^{\theta, t, x} - \xi) | \mathcal{F}_t), \quad t \leq T, \quad x \in \mathbb{R}, \quad (5.3)$$

for the exponential utility function $u(x) := -\exp(-\alpha x)$ with absolute risk aversion $\alpha > 0$, with some additive liability ξ and for wealth processes $X^{\theta, t, x}$ of admissible trading strategies θ as defined below. We are going to show, how the value process v and optimal trading strategy θ^* for the problem (5.3) can be fully described by a JBSDEs for two distinct problem cases.

5.1.1 Case with continuous price processes of risky assets

The set of available trading strategies Θ consists of all \mathbb{R}^d -valued, predictable, S -integrable processes θ for which the following two conditions are satisfied: $\mathbb{E}(\int_0^T |\theta_t|^2 dt)$ is finite, and the family $\{\exp(-\alpha \int_0^\tau \theta_t d\widehat{B}_t) \mid \tau \text{ stopping time, } \tau \leq T\}$ of random variables is uniformly integrable under \mathbb{P} . Starting from initial capital $x \in \mathbb{R}$ at some time $t \leq T$, the wealth process corresponding to investment strategy $\theta \in \Theta$ is given by $X_s^\theta = X_s^{\theta, t, x} = x + \int_t^s \theta_u d\widehat{B}_u$, $s \in [t, T]$.

For this subsection, we assume $k = d$ (so f will not be quadratic in z). Let (Y, Z, U) in $\mathcal{S}_{\mathbb{P}}^\infty \times \mathcal{L}_{\mathbb{P}}^2(\widehat{B}) \times \mathcal{L}_{\mathbb{P}}^2(\widetilde{\mu})$ be the unique solution to the BSDE $Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s) ds - \int_t^T Z_s d\widehat{B}_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(ds, de)$ under the minimal local martingale measure $\widehat{\mathbb{P}}$ for the generator

$$f_t(y, z, u) := -\frac{|\varphi_t|^2}{2\alpha} + \int_E \frac{\exp(\alpha u(e)) - \alpha u(e) - 1}{\alpha} \zeta(t, e) \lambda(de) \quad (5.4)$$

which does exist by Theorem 4.13. Under \mathbb{P} the BSDE is of the form

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s) - \varphi_s Z_s ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(e) \widetilde{\mu}(ds, de).$$

To prove optimality by a martingale principle one constructs, cf. [HIM05, Sek06], a family of processes $(V^\theta)_{\theta \in \Theta}$ such that three conditions are satisfied: (i) $V_t^\theta = V_t$ is a fixed \mathcal{F}_t -measurable bounded random variable invariant over $\theta \in \Theta$, (ii) $V_T^\theta = -\exp(-\alpha(X_T^\theta - \xi)) = -\exp(-\alpha(x + \int_t^T \theta_s d\widehat{B}_s - \xi))$, and (iii) V^θ is a supermartingale for all $\theta \in \Theta$ and there exists a $\theta^* \in \Theta$ such that $V_s^{\theta^*}$ ($s \in [t, T]$) is a \mathbb{P} -martingale. Then θ^* is the optimal strategy and $(V_s^{\theta^*})_{s \in [t, T]}$ is the value process of the control problem (5.3). Indeed, $\mathbb{E}(V_T^\theta | \mathcal{F}_t) \leq V_t^\theta = V_t^{\theta^*} = \mathbb{E}(V_T^{\theta^*} | \mathcal{F}_t)$ for each $\theta \in \Theta$ implies $v_t(x) = \text{ess sup}_{\theta \in \Theta} \mathbb{E}(V_T^\theta | \mathcal{F}_t) = V_t^{\theta^*}$. An ansatz $V^\theta = u(X^\theta - Y)$ yields

$$\begin{aligned} V_s^\theta &= V_t^\theta \exp\left(\frac{\alpha^2}{2} \int_t^s \left|\theta_r - Z_r - \frac{\varphi_r}{\alpha}\right|^2 dr\right) \mathcal{E}(M)_t^s \quad \text{for all } s \in [t, T], \quad \text{with} \\ M_t &= -\alpha \int_0^t \theta_r - Z_r d\widehat{B}_r + \int_0^t \int_E \exp(\alpha U_r(e) - 1) \widetilde{\mu}(dr, de) \quad \text{and} \quad \mathcal{E}(M)_t^s := \frac{\mathcal{E}(M)_s}{\mathcal{E}(M)_t}. \end{aligned}$$

Therefore, V^θ is a supermartingale for all $\theta \in \Theta$ and a martingale for $\theta^* = Z + \varphi/\alpha$ due to the fact that $\mathcal{E}(M)$ is a (local) martingale of the form

$$\mathcal{E}(M)_s = \exp\left(-\frac{\alpha^2}{2} \int_0^s |\theta_u - Z_u - \varphi_u/\alpha|^2 du\right) \exp\left(-\alpha\left(Y_0 + \int_0^s \theta_u d\widehat{B}_u - Y_s\right)\right).$$

Using the boundedness of Y , one readily obtains by arguments like in [HIM05, Mor10] that $\mathcal{E}(M)$ is uniformly integrable and hence a martingale (see e.g. eqn. (4.19) in [Bec06]). This yields

Example 5.1. Let $k = d$ and $\lambda(E) \leq \infty$. Let $(Y, Z, U) \in \mathcal{S}_{\mathbb{P}}^\infty \times \mathcal{L}_{\mathbb{P}}^2(\widehat{B}) \times \mathcal{L}_{\mathbb{P}}^2(\widetilde{\mu})$ be the unique solution to the BSDE (ξ, f) under $\widehat{\mathbb{P}}$ for generator f from (5.4). Then the strategy $\theta^* = Z + \varphi/\alpha$ is optimal for the control problem (5.3) and achieves at any time $t \leq T$ the maximal expected exponential utility $v_t(x) = -\exp(-\alpha(x - Y_t)) = V_t^{\theta^*}$.

Note that exponential utility maximization is closely related to the entropic convex risk measure. We will relate to this in Example 5.3 to further illustrate the scope of our results.

5.1.2 Case with discontinuous risky asset price processes

We further illustrate the extend to which results by [Mor09, Mor10], who has pioneered the stability approach to BSDE with jumps specifically for exponential utility, fit into our framework and demonstrate by concrete examples some notable differences in scope in relation to complementary approaches, like [KTPZ15, LS14]. Let us consider the same utility problem but now in a financial market with pure-jump asset price processes, possibly of infinite activity (as e.g. in the CGMY model of [CGM⁺02]), and with constraint on trading strategies. We note that a pure-jump setting

appears as a natural setup for our JBSDE results, which admit for generators that are (roughly said) 'quadratic' in u but not in z , differently from [Mor10, KTPZ15, LS14].

Let $\mu = \mu^L$ be the random measure associated to a pure-jump Lévy process L with Lévy measure $\lambda(\mathrm{d}e)$, on $E = \mathbb{R}^1 \setminus \{0\}$. Let $\mathbb{F} = \mathbb{F}^L$ be the usual filtration generated by L . The compensated random measure $\tilde{\mu} = \tilde{\mu}^L := \mu^L - \nu$, with $\nu(\mathrm{d}t, \mathrm{d}e) = \lambda(\mathrm{d}e)\mathrm{d}t$ of L alone has the weak PRP w.r.t. the filtration \mathbb{F} (see Example 2.1.1.). Note that μ could be of infinite activity, i.e. $\lambda(E) \leq \infty$, for instance for L being a Gamma process. In contrast to the setup of Section 5.1.1, we consider now a financial market whose single risky asset prices evolves in a non-continuous fashion, being given by a pure-jump process

$$\mathrm{d}S_t = S_{t-} \left(\beta_t \mathrm{d}t + \int_E \psi_t(e) \tilde{\mu}(\mathrm{d}t, \mathrm{d}e) \right) \quad \text{for } t \in [0, T], \text{ with } S_0 \in (0, \infty),$$

where β is predictable and bounded, and $\psi > -1$ is $\tilde{\mathcal{P}}$ -measurable, in $L^2(\mathbb{P} \otimes \lambda \otimes \mathrm{d}t) \cap L^\infty(\mathbb{P} \otimes \lambda \otimes \mathrm{d}t)$ and satisfies $\int_E |\psi_t(e)|^2 \lambda(\mathrm{d}e) < \text{const.}$ $\mathbb{P} \otimes \mathrm{d}t$ -a.e.. The set Θ of admissible trading strategies consists of all \mathbb{R} -valued predictable S -integrable processes $\theta \in L^2(\mathbb{P} \otimes \mathrm{d}t)$, such that $\theta_t(\omega) \in C$ for all (t, ω) , for a fixed compact set $C \subset \mathbb{R}$ of trading constraint containing 0. Interpreting trading strategies θ as amount of wealth invested into the risky asset yields wealth process $X^{\theta, t, x}$ from initial capital x at time t as

$$X_s^{\theta, t, x} = X_t^{\theta, t, x} + \int_t^s \theta_u \frac{\mathrm{d}S_u}{S_{u-}} = x + \int_t^s \theta_u \left(\beta_u \mathrm{d}u + \int_E \psi_u(e) \tilde{\mu}(\mathrm{d}u, \mathrm{d}e) \right), \quad s \geq t.$$

Because of the compactness of C and the fact that $\psi \in L^2(\mathbb{P} \otimes \lambda \otimes \mathrm{d}t) \cap L^\infty(\mathbb{P} \otimes \lambda \otimes \mathrm{d}t)$, admissible strategies are bounded and for all $\theta \in \Theta$ one can verify that $\{\exp(-\alpha X_\tau^\theta) \mid \tau \text{ an } \mathbb{F}\text{-stopping time}\}$ is uniformly integrable; arguments being like in [Mor10, Lem.1]. Consider the JBSDE

$$-\mathrm{d}Y_t = f(t, U_t) \mathrm{d}t - \int_E U_t(e) \tilde{\mu}(\mathrm{d}t, \mathrm{d}e), \quad Y_T = \xi, \quad (5.5)$$

with terminal condition $\xi \in L^\infty(\mathcal{F}_T)$ and generator f defined pointwise by

$$f(t, u) := \inf_{\theta \in C} \left(-\theta \beta_t + \int_E g_\alpha(u(e) - \theta \psi_t(e)) \lambda(\mathrm{d}e) \right), \quad t \in [0, T], \quad (5.6)$$

for the function $g_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ with $g_\alpha(u) := (e^{\alpha u} - \alpha u - 1)/\alpha$. We have the following

Example 5.2. Let $(Y, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(\tilde{\mu})$ be the unique solution to the JBSDE (5.5). Then the strategy θ^* such that θ_t^* achieves the infimum in (5.6) for $f(t, U_t)$ is optimal for the control problem (5.3) and achieves at any $t \in [0, T]$ the maximal expected exponential utility $v_t(x) = -\exp(-\alpha(x - Y_t)) = V_t^{\theta^*}$.

Proof. Using the martingale optimality principle one obtains, like in the cited literature and analogously to Section 5.1.1, that if $(Y, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(\tilde{\mu})$ is a solution to the JBSDE (5.5) then the solution to the utility maximization problem (5.3) is indeed given by $v_t(x) = u(x - Y_t)$, with the strategy θ^* where $\theta_t^*(\omega)$ achieves the infimum $f(\omega, t, U_t(\omega))$ in (5.6) for all (ω, t) being optimal (it exists by measurable selection [Roc76]). To complete the derivation of this example, it thus just remains to show that the JBSDE (5.5) indeed admits a unique solution, with trivial Z -component $Z = 0$. This is shown by applying Theorem 4.11 and Corollary 4.12 since $\xi \in L^\infty(\mathcal{F}_T^L)$ and the generator f does not depend on the z -argument and is \mathbb{F}^L -predictable in (t, ω) . It is straightforward, albeit somewhat tedious, to verify that the conditions 1-5 and (B_{γ^n}) , $n \in \mathbb{N}$, for Theorem 4.11 are indeed satisfied for the sequence of \mathbb{F}^L -predictable generators functions

$$f^n(t, u) := \inf_{\theta \in C} \left(-\theta \beta_t + \int_{A_n} g_\alpha(u(e) - \theta \psi_t(e)) \lambda(\mathrm{d}e) \right),$$

where $(A_n)_n$ is a sequence of measurable sets with $A_n \uparrow E$ and $\lambda(A_n) < \infty$ for all $n \in \mathbb{N}$, typically $A_n = (-\infty, -1/n] \cup [1/n, +\infty)$. Let us refer to [Ken15, Example 1.32] for details of this verification, but explain here how to proceed further with the proof.

By the first claim of Theorem 4.11 (together with Corollary 4.12) one then gets existence of a solution $(Y, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(\tilde{\mu})$ to the JBSDE (5.5), such that $U * \tilde{\mu}$ is a BMO-martingale. To obtain uniqueness by applying the second claim, we need to check that f satisfies condition (\mathbf{A}'_γ) : To this end, we define $\gamma_t^{u,u'}(e) := \sup_{\theta \in C} \gamma_t^{\theta,u,u'}(e) \mathbb{1}_{\{u \geq u'\}} + \inf_{\theta \in C} \gamma_t^{\theta,u,u'}(e) \mathbb{1}_{\{u < u'\}}$, for $\gamma_t^{\theta,u,u'}(e) := \int_0^1 g'_\alpha(l(u - \theta\psi_t(e)) + (1-l)(u' - \theta\psi_t(e)))dl$. Then (by Examples 3.5 and 3.8-2.) we get $f(t, U_t) - f(t, U'_t) \leq \int_E \gamma_t^{U,U'}(e)(U_t(e) - U'_t(e))\lambda(de)$ for all U, U' with $|U|_\infty < \infty, |U'|_\infty < \infty$. Now let u, u' be bounded by $c > 0$; Since $g'_\alpha(0) = 0$, applying the mean-value theorem to g'_α in the expression of $\gamma_t^{\theta,u,u'}$ gives $|\gamma_t^{\theta,u,u'}(e)| \leq \sup_{|x| \leq \tilde{c}} |g''_\alpha(x)|(|u| + |u'| + |\theta||\psi_t(e)|)$ for all $\theta \in C$, where $\tilde{c} := c + \|\psi\|_\infty \text{diam}(C)$. This implies (for $c = |U|_\infty \vee |U'|_\infty < \infty$)

$$\sup_{\theta \in C} |\gamma_t^{\theta,U,U'}(e)| \leq \sup_{|x| \leq \tilde{c}} |g''_\alpha(x)| \left(|U_t(e)| + |U'_t(e)| + \text{diam}(C)|\psi_t(e)| \right).$$

Since $|\inf_\theta \gamma^\theta| \leq \sup_\theta |\gamma^\theta|$, $|\sup_\theta \gamma^\theta| \leq \sup_\theta |\gamma^\theta|$ and $\psi * \tilde{\mu}$ is a BMO-martingale (as ψ is bounded and $\int_E |\psi_t(e)|^2 \lambda(de) < \text{const.}$, $\mathbb{P} \otimes dt$ -a.e. by assumption), then $\gamma^{U,U'} * \tilde{\mu}$ is a BMO-martingale if $U * \tilde{\mu}$ and $U' * \tilde{\mu}$ are, thanks to $|U|_\infty < \infty$ and $|U'|_\infty < \infty$. Hence f satisfies (\mathbf{A}'_γ) . \square

Example 5.3. (entropic convex risk measure) Let us consider the special case $\beta = \psi \equiv 0$ and $S \equiv 1$, i.e. without trading opportunities in a risky asset. One gets the well-known (dynamic) entropic risk measure, whose respective JBSDE solution $Y_t = (1/\alpha) \log \mathbb{E}(\exp(\alpha\xi)|\mathcal{F}_t)$ can be identified directly by exponential transformation. In the setup of the present subsection, this JBSDE is covered by the application study of [Mor10] and also by our comparison and wellposedness theorems, without any further conditions on the pure-jump Lévy process. In contrast, let us demonstrate that the same scope is not already offered by the seminal comparison Thm.2.5 of [Roy06] because her key condition (\mathbf{A}_γ) is not satisfied, which is also supposed [KTPZ15, as Assumpt.6.1 for wellposedness in Thm.6.3(i) and for comparison in Prop.6.4]: Indeed for $\alpha := 1$, consider a compound Poisson process L (being of finite activity) with uniformly distributed jump heights, taking $\lambda(dx) := \mathbb{1}_{(0,1]}dx$ ($x \in E = \mathbb{R} \setminus \{0\}$). Clearly, the generator $f(u) = \int_E \exp(u(x)) - u(x) - 1 \lambda(dx) =: \int_E g(u(x)) \lambda(dx)$ is not Lipschitz in $u \in L^2(\lambda)$. With $u^\pm(x) := (1/2)(\pm x^{-3/2} + nx) \mathbb{1}_{(1/n,1]}(x)$ in $L^2(\lambda)$ for $n \in \mathbb{N}$, we get $\int_0^1 e^{u^+} - e^{u^-} - u^+ + u^- d\lambda \rightarrow \infty$ for $n \rightarrow \infty$ while $\int_0^1 (u^+ - u^-)(1 \wedge |x|) dx \leq \int_0^1 x^{-1/2} dx < \infty$ for all n , noting that $e^{u^+} - e^{u^-} - u^+ + u^- \geq nx^{-1/2} \mathbb{1}_{(1/n,1]}$. Thus, there is no constants $c_1 \in (-1, 0]$, $c_2 < \infty$, such that $f(u) - f(v) \leq \int_E (u - v)(x) \gamma^{u,v}(x) \lambda(dx)$ for all u, v , with $c_1(1 \wedge |x|) \leq \gamma^{u,v}(x) \leq c_2(1 \wedge |x|)$; So condition (\mathbf{A}_γ) in [Roy06], which is used as Assumption 6.1 in [KTPZ15], is violated here. Actually, the (\mathbf{A}_γ) condition in [Roy06] implies Lipschitz continuity in u .

Example 5.4. We continue with the previous entropic risk example, but now take a standard Poisson process instead, i.e. $\lambda(dx) = \delta_{\{1\}}(dx)$ as Dirac point measure at the fixed jump height 1. Then $L^2(\lambda)$ is isomorphic to \mathbb{R} , and $f(u) = g(u) = \exp(u) - u - 1$ for $u \in \mathbb{R}$. Obviously g' and g'' are of exponential growth (in u) and cannot be bounded globally in $u \in \mathbb{R}$ by an affine function or by constants; hence Assumption 5.1(iii) for [KTPZ15, Thm.5.4] cannot be satisfied. Moreover, also Assumption 4.3(iii) for [KTPZ15, Thms.4.3, 5.4 and 6.3(ii)], noting their Lem.5.4, appears clearly violated since (taking $u' = 0$) there exist no $\psi, c \in \mathbb{R}$ such that $|g(u) - \psi u| \leq c|u|^2$ for all $u \in \mathbb{R}$.

Let us emphasize that, since the function $(u, \theta) \mapsto f^\theta(\cdot, u) := -\theta\beta + \int_E g_\alpha(u(e) - \theta\psi_t(e))\lambda(de)$ is convex, the generator constructed as $f = \inf_{\theta \in C} f^\theta(\cdot, u)$ (cf. 5.6) would be convex in u if the constraint set C were assumed to be convex; But for non-convex trading constraints C the generator can be non-convex in general. Similar constructions of generators are typical in this application context see e.g. [LS14, eqn. (15)]. There are interesting results in the literature (e.g. [LS14, Thm.A.28, Cor.A.29, used in proofs for Thms.4.3, 4.5] or [KTPZ15, Thm.6.3(ii), Prop.6.4(ii)]) on JBSDEs that require convexity of the generator function in order to prove wellposedness or comparison. Relying on the monotone stability approach, our results do not require such a property of the generator. Next, we give a concrete example where f in (5.6) for the (primal) control problem is non-convex in u , which is not covered by results for wellposedness and comparison of JBSDE as in [LS14, KTPZ15, KTPZ16]. To this end, consider a simple trading constraint that is non-convex, taking $C := \{\theta^0, \dots, \theta^m\} \subset \mathbb{R}$ as a finite set including $\theta^0 := 0$ zero. Here f of JBSDE (5.5) becomes

$$f(t, u) = \inf_{k \in \{0, \dots, m\}} \left(-\theta^k \beta_t + \int_E g_\alpha(u(e) - \theta^k \psi_t(e)) \lambda(de) \right).$$

Example 5.5. Continuing with the above f , now let us take the particular simple case where $\lambda(\text{de}) = \delta_{\{1\}}(\text{de})$, i.e. L is a standard Poisson process with constant jump height 1, and let $\alpha = 1$ and $\beta = 0$. Observing that $L^2(\lambda)$ here is isomorphic to \mathbb{R} , we obtain

$$f(t, u) = \min_{k \in \{1, \dots, m\}} \left(e^{(u - \theta^k \psi_t)} - (u - \theta^k \psi_t) - 1 \right), \quad (5.7)$$

which is readily seen to be non-convex in $u \in \mathbb{R}$ unless $\psi \equiv 0$ or $C = \{0\}$. However f in (5.7) is absolutely continuous in u with density function being strictly greater than -1 , locally bounded in u from above and locally of linear growth; as this f satisfies condition $(\mathbf{A}_{\text{inf}})$, existence and uniqueness for the corresponding JBSDE (5.5) can be readily obtained by applying Theorem 4.13.

Moreover one can check again, similarly as before, that in this example the assumptions of [Roy06, Thm.2.5] and [KTPZ15, Thm.5.4, Thm.6.3(i) and (ii), and Prop.6.4] for comparison and wellposedness of JBSDE are not satisfied; Neither are the assumptions for the JBSDE results of [LS14, Thm.A28, Cor.A29, Prop.A30, and the proofs for Thms.4.3 and 4.5 – all involving convexity assumption ‘(c)’] satisfied, because the generator of this example is clearly non-convex.

While the above examples demonstrate the different scope of our results in relation to [Roy06, LS14, KTPZ15], let us proceed with further examples which are clearly out of scope of the study by [Mor09, Mor10] on exponential utility application by using the monotone stability approach.

5.2 Power utility maximization

Again for the market with stock price dynamics (5.1), we consider the utility maximization problem

$$v_t(x) = \text{ess sup}_{\theta \in \Theta} \mathbb{E}(u(X_T^{\theta, t, x}) \xi \mid \mathcal{F}_t) = \frac{1}{\gamma} \text{ess sup}_{\theta \in \Theta} \mathbb{E}(u(X_T^{\theta, t, x} \xi') \mid \mathcal{F}_t), \quad t \leq T, x > 0, \quad (5.8)$$

for power utility $u(x) = x^\gamma / \gamma$ with relative risk aversion $1 - \gamma > 0$ for $\gamma \in (0, 1)$, with multiplicative liability ξ ($\xi' := (\gamma \xi)^{1/\gamma}$ being, e.g., an unknown future tax rate). We parametrize strategies θ by fractions of wealth invested. Then the wealth process is $X_s^\theta = X_s^{\theta, t, x} = x + \int_t^s X_u^\theta \theta_u d\widehat{B}_u = x \mathcal{E}(\int_t^s \theta d\widehat{B})_s^\theta$ for $s \in [t, T]$, for $\theta \in \Theta$, with the set Θ of strategies given by all \mathbb{R}^d -valued, predictable, S -integrable processes such that $\theta \bullet B$ is a BMO(\mathbb{P})-martingale, cf. [HWY92].

Proposition 5.6. Let $k = d$. Assume that there is a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ to the BSDE (ξ, f) with $f_t(y, z, u) := (\gamma/(2 - 2\gamma)) y |\varphi_t + y/z|^2$ and $\int Z dB \in \text{BMO}(\mathbb{P})$ and where ξ is in $L^\infty(\mathcal{F}_T)$ with $\xi \geq c$ for some $c > 0$. Then $Y \geq c$ holds and $V^\theta := u(X^\theta)Y$ is a supermartingale for all θ in Θ and V^{θ^*} is a martingale for $\theta^* := (1 - \gamma)^{-1}(\varphi + Z/Y_-) \in \Theta$.

Proof. Clearly, V^θ is adapted. Kazamaki’s criterion $\mathcal{E}(\int_0^\cdot \gamma \theta_u d\widehat{B}_u)$ is an r -integrable martingale for some $r > 1$. Hence $\sup_{t \leq s \leq T} \mathcal{E}(\int_0^s \gamma \theta_u d\widehat{B}_u)_t^\theta$ is integrable by Doob’s inequality. By

$$\mathcal{E}(\theta \bullet \widehat{B})^\gamma = \mathcal{E}(\gamma \theta \bullet \widehat{B}) \exp \left(-\frac{1}{2} \gamma (1 - \gamma) \int_0^\cdot |\theta_u|^2 du \right) \leq \mathcal{E}(\gamma \theta \bullet \widehat{B}),$$

we conclude that V^θ is dominated by $\sup_{t \leq s \leq T} U(X_s^\theta) |Y|_\infty \in L^1(\mathbb{P})$. By Itô’s formula, dV_s^θ equals a local martingale plus the finite variation part

$$u(X_s^\theta) \left(-f_s(Y_{s-}, Z_s, U_s) + \gamma \left(Y_{s-} \left(\theta_s \varphi_s + \frac{1}{2} (\gamma - 1) |\theta_s|^2 \right) + \theta_s Z_s \right) \right) ds.$$

The latter part is decreasing for all $\theta \in \Theta$ and vanishes at zero for $\theta = \theta^*$. So V^θ is a local (super)martingale. Uniform integrability of V^θ yields the (super)martingale property. By the classical martingale optimality principle of optimal control follows that $v_t(x) = \text{ess sup}_{\theta \in \Theta} \mathbb{E}(u(X_T^\theta \xi^{1/\gamma}) \mid \mathcal{F}_t)$ equals $V_t^{\theta^*} = \gamma^{-1} x^\gamma Y_t$, and evaluating at $\theta \equiv 0$ yields $\gamma^{-1} x^\gamma \mathbb{E}(\xi \mid \mathcal{F}_t) \leq \gamma^{-1} x^\gamma Y_t$ and hence $Y \geq c$. Note that θ^* is in Θ since φ is bounded, Y is bounded away from 0 and Z is an BMO integrand. \square

Let (Y, Z, U) be a solution to the BSDE (ξ, f) with the above data. Since a suitable solution theory for quadratic BSDEs with jumps is not available, we transform coordinates by letting

$$\widetilde{Y}_t := Y_t^{\frac{1}{1-\gamma}}, \quad \widetilde{Z}_t := \frac{1}{1-\gamma} Y_{t-}^{\frac{\gamma}{1-\gamma}} Z_t \quad \text{and} \quad \widetilde{U}_t := (Y_{t-} + U_t)^{\frac{1}{1-\gamma}} - Y_{t-}^{\frac{1}{1-\gamma}}, \quad (5.9)$$

such that $(\tilde{Y}, \tilde{Z}, \tilde{U})$ solves the BSDE for data $(\tilde{\xi}, \tilde{f})$ with $\tilde{\xi} = \xi^{1/(1-\gamma)}$ and $\tilde{f}_t(y, z, u)$ given by

$$\frac{\gamma|\varphi_t|^2}{2(1-\gamma)^2}y + \frac{\gamma}{1-\gamma}\varphi_t z + \int_E \left(\frac{1}{1-\gamma} ((u(e) + y)^{1-\gamma}y^\gamma - y) - u(e) \right) \zeta(t, e) \lambda(de).$$

Looking at the proof of Lemma 2.2, we may assume that $U + Y_-$ coincides pointwise with Y_- or Y so that the above transformation is well-defined due to $Y \geq c$. In fact, (5.9) gives a bijection between solutions with positive Y -components to the BSDEs (ξ, f) and $(\tilde{\xi}, \tilde{f})$ in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$.

Next, we show the existence of a JBSDE solution for data (ξ, f) with $\xi \geq c$ for some $c > 0$. Under the probability measure $d\tilde{\mathbb{P}} := \mathcal{E}(\gamma(1-\gamma)^{-1}\varphi \bullet B)_T d\mathbb{P}$ the process $\tilde{B} = B - \int_0^\cdot \gamma(1-\gamma)^{-1}\varphi_t dt$ is a Brownian motion and the JBSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}_s(\tilde{Y}_{s-}, \tilde{Z}_s, \tilde{U}_s) ds - \int_t^T \tilde{Z}_s d\tilde{B}_s - \int_t^T \int_E \tilde{U}_s(e) \tilde{\mu}(ds, de)$$

under \mathbb{P} is of the following form under $\tilde{\mathbb{P}}$

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \left(\tilde{f}_s(\tilde{Y}_{s-}, \tilde{Z}_s, \tilde{U}_s) - \frac{\gamma\varphi_s}{1-\gamma}\tilde{Z}_s \right) ds - \int_t^T \tilde{Z}_s d\tilde{B}_s - \int_t^T \int_E \tilde{U}_s(e) \tilde{\mu}(ds, de), \quad (5.10)$$

noting that ν is the compensator of μ under \mathbb{P} and $\tilde{\mathbb{P}}$ as well. In fact, we have

Lemma 5.7. *Assume $\lambda(E) < \infty$. Then $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ solves the BSDE $(\tilde{\xi}, \tilde{f})$ such that $\int \tilde{Z} d\tilde{B}$ is in $\text{BMO}(\tilde{\mathbb{P}})$ if and only if $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{S}_{\mathbb{P}}^\infty \times \mathcal{L}_{\mathbb{P}}^2(\tilde{B}) \times \mathcal{L}_{\mathbb{P}}^2(\tilde{\mu})$ solves the BSDE $(\tilde{\xi}, \tilde{f}(y, z, u) - \gamma(1-\gamma)^{-1}\varphi z)$ such that $\int \tilde{Z} d\tilde{B}$ is in $\text{BMO}(\tilde{\mathbb{P}})$.*

Proof. Equivalence of \mathbb{P} and $\tilde{\mathbb{P}}$ imply that $\tilde{Y} \in \mathcal{S}^\infty$ if and only if (iff) $\tilde{Y} \in \mathcal{S}_{\tilde{\mathbb{P}}}^\infty$. Assuming that $\lambda(E) < \infty$, $\tilde{U} \in \mathcal{L}^2(\tilde{\mu})$ iff $\tilde{U} \in \mathcal{L}_{\tilde{\mathbb{P}}}^2(\tilde{\mu})$ due to the boundedness of \tilde{U} . By [Kaz94, Thm.3.6], the restriction of the Girsanov transform $\Phi : \mathcal{M}_c^{\text{loc},0}(\mathbb{P}) \longrightarrow \mathcal{M}_c^{\text{loc},0}(\tilde{\mathbb{P}})$, with $M \mapsto M - \langle M, \int_0^\cdot \frac{\gamma\varphi}{1-\gamma} dB_s \rangle$, onto $\text{BMO}(\mathbb{P})$ yields a bijection between $\text{BMO}(\mathbb{P})$ -martingales and $\text{BMO}(\tilde{\mathbb{P}})$ -martingales. Consequently, $\int \tilde{Z} d\tilde{B}$ is in $\text{BMO}(\mathbb{P})$ iff $\int \tilde{Z} d\tilde{B}$ is in $\text{BMO}(\tilde{\mathbb{P}})$ for $Z = (1-\gamma)\tilde{Y}^\gamma \tilde{Z}$ since $\Phi(\int \tilde{Z} d\tilde{B}) = \int \tilde{Z} d\tilde{B} - \int \gamma(1-\gamma)^{-1}\varphi \tilde{Z}_s ds = \int \tilde{Z} d\tilde{B}$. In particular, $\tilde{Z} \in \mathcal{L}^2(B)$ iff $\tilde{Z} \in \mathcal{L}_{\tilde{\mathbb{P}}}^2(\tilde{B})$. \square

According to Corollary 4.6 there exists a unique solution $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{S}_{\tilde{\mathbb{P}}}^\infty \times \mathcal{L}_{\tilde{\mathbb{P}}}^2(B) \times \mathcal{L}_{\tilde{\mathbb{P}}}^2(\tilde{\mu})$ with positive Y -component to the BSDE (5.10) with

$$c^{\frac{1}{1-\gamma}} \exp\left(-\frac{\gamma|\varphi|_\infty^2}{2(1-\gamma)^2}(T-t)\right) \leq \tilde{Y}_t \leq |\xi|_\infty \exp\left(\frac{\gamma|\varphi|_\infty^2}{2(1-\gamma)^2}(T-t)\right)$$

such that $\int \tilde{Z} d\tilde{B}$ and $\tilde{U} * \tilde{\mu}^{\tilde{\mathbb{P}}}$ are $\text{BMO}(\tilde{\mathbb{P}})$ -martingales. By Lemma 5.7 and the statement of Proposition 5.6 that every bounded solution to the BSDE (ξ, f) is bounded from below away from zero in $Y \geq c > 0$, there is a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ with $\int Z d\tilde{B} \in \text{BMO}(\mathbb{P})$ and it is given by the coordinate transform (5.9). We note that Y (resp. \tilde{Y}) can be interpreted as (dual) opportunity process, see [Nut10, Sect.4]. Overall, we obtain

Theorem 5.8. *Assume $\lambda(E) < \infty$ and $d = k$. Let $f_s(y, z, u) = \gamma(2-2\gamma)^{-1}y|\varphi_s + z/y|^2$ and let $\xi \in L^\infty(\mathcal{F}_T)$ with $\xi \geq c$ for some $c > 0$. Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ with $\int Z d\tilde{B} \in \text{BMO}(\mathbb{P})$ to the BSDE (ξ, f) . Then the strategy $\theta_s^* = (1-\gamma)^{-1}(\varphi_s + Z_s/Y_{s-})$ is optimal for the control problem (5.8), achieving $v_t(x) = \gamma^{-1}x^\gamma Y_t = V_t^{\theta^*}$.*

5.3 Valuation in incomplete markets by no-good-deal bounds

In incomplete financial markets without arbitrage, there exist infinitely many pricing measures and the bounds imposed on valuations solely by the principle of no-arbitrage are typically far too wide for applications in practice. The concept of good-deal bounds [CSR00], widely acclaimed in the

finance literature, has been introduced to obtain tighter bounds, by ruling out not only arbitrage opportunities but also trading opportunities that are “too favorable”, so-called good deals. By admitting as pricing measures only a subset (with suitable economic meaning) of all equivalent local martingale measures, called the no-good-deal pricing measures, one arrives at the tighter (no) good-deal bounds for valuation. In [CSR00, BS06] good deals have been defined in terms of too favorable instantaneous Sharpe ratios (of the rate of excess return per unit rate of variance risk). The continuous time analysis for diffusion processes [CSR00] has been generalized in [BS06] to a (Markovian) jump-diffusion setup for price and factor processes which can have jumps. See [DP15] for a study of a case where the measure λ has finite support.

In our setting, the following description of the set \mathcal{M}^e of martingale measures is routine.

Proposition 5.9. *\mathcal{M}^e consists of those measures $\mathbb{Q} \approx \mathbb{P}$ such that $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(\beta \bullet B + \gamma * \tilde{\mu})$, where $\gamma > -1$ is a $\tilde{\mathcal{P}}$ -measurable and $\tilde{\mu}$ -integrable function, and β is a predictable process with $\int_0^T |\beta_s|^2 ds < \infty$, satisfying $\beta = -\varphi + \eta$, such that $\eta \in \text{Ker } \sigma$, $\mathbb{P} \otimes dt$ -a.e..*

We will refer to the tuple (γ, β) for such a density $d\mathbb{Q}/d\mathbb{P}$ as the Girsanov kernel of \mathbb{Q} relative to \mathbb{P} . Clearly, our market is incomplete in general as there exists infinitely many measures in \mathcal{M}^e if $\tilde{\mu}$ is non-trivial or $k < d$. Björk and Slinko employed an extended Hansen-Jagannathan inequality [BS06, see Sect.2] to bound the instantaneous Sharpe ratio by imposing a bound on market prices of risk. More precisely, Thm.2.3 of [BS06] showed that the instantaneous Sharpe ratio SR_t in any market extended by derivative assets (i.e. for any local \mathbb{Q} -martingale) at any time t satisfies $|SR_t| \leq \|(\gamma_t, \beta_t)\|_{L^2(\lambda_t) \times \mathbb{R}^d}$, with a (sharp) upper bound in terms of an L^2 -norm for Girsanov kernels (γ, β) of pricing measures in \mathcal{M}^e , with $\lambda_t(\omega)(de) := \zeta_t(\omega, e)\lambda(de)$. As no-good-deal restriction they therefore impose a bound on the kernels of pricing measures

$$\|(\gamma_t, \beta_t)\|_{L^2(\lambda_t) \times \mathbb{R}^d}^2 = \|\gamma_t\|_{L^2(\lambda_t)}^2 + |\beta_t|_{\mathbb{R}^d}^2 \leq K^2, \quad t \leq T, \quad (5.11)$$

by some given constant $K > 0$. In a Markovian jump-diffusion model for asset prices plus additional factor processes, they describe good-deal bounds as solutions of nonlinear partial-integro differential equations, using (formal) HJB dynamic programming techniques. To complement the analysis of the problem as posed by [BS06], we are going to describe the dynamic good deal bounds rigorously by JBSEs in a more general, possibly non-Markovian, setting with no-good-deal restriction of the same type (5.11) where K can be a positive predictable bounded process instead of a constant.

To this end, let the correspondence (set-valued) process C be given by

$$C_t := \left\{ (\gamma, \eta) \in L^2(\lambda_t) \times \mathbb{R}^d \mid \gamma > -1, \eta \in \text{Ker } \sigma_t, \text{ and } \|\gamma\|_{L^2(\lambda_t)}^2 + |\eta|_{\mathbb{R}^d}^2 + |\varphi_t|_{\mathbb{R}^d}^2 \leq K_t^2 \right\}. \quad (5.12)$$

We will write $(\gamma, \eta) \in C$ to denote that η is a predictable process and γ is a $\tilde{\mathcal{P}}$ -measurable process with $(\gamma_t(\omega), \eta_t(\omega)) \in C_t(\omega)$ holding for all $(t, \omega) \in [0, T] \times \Omega$. For $(\gamma, \eta) \in C$, we know (cf. Example 3.3.1) that $\mathcal{E}((-\varphi + \eta) \bullet B + \gamma * \tilde{\mu}) > 0$ is a martingale that defines as a density process of a probability measure $\mathbb{Q}^{\gamma, \eta}$ which is equivalent to \mathbb{P} . The set of such probability measures

$$\mathcal{Q}^{\text{ngd}} := \{\mathbb{Q}^{\gamma, \eta} \mid (\gamma, \eta) \in C\} \subseteq \mathcal{M}^e, \quad (5.13)$$

defines our set of no-good-deal measures. Beyond boundedness of φ , assume that $|\varphi_t|_{\mathbb{R}^d} + \epsilon < K_t$ holds for for some $\epsilon > 0$ for all $t \leq T$. Then, in particular, the minimal martingale measure $\hat{\mathbb{P}} = \mathbb{Q}^{\hat{\gamma}, \hat{\eta}}$ is in $\mathcal{Q}^{\text{ngd}} \neq \emptyset$, with $(\hat{\gamma}, \hat{\eta}) \equiv (0, 0) \in C$. For contingent claims $X \in L^\infty(\mathbb{P})$, the processes

$$\pi_t^u(X) := \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t) \quad \text{and} \quad \pi_t^l(X) := \text{ess inf}_{\mathbb{Q} \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t), \quad t \leq T,$$

define the upper and lower good-deal bounds. Noting $\pi_t^l(X) = -\pi_t^u(-X)$, we focus on $\pi_t^u(-X)$. One can check that the sets \mathcal{Q}^{ngd} defined in (5.13) and \mathcal{M}^e are convex and multiplicatively stable (see [Del06]), ensuring good dynamic properties of the good-deal bounds, e.g. time-consistency and recursiveness (cf. e.g. [Bec09, Prop.2.6]). By applying the comparison result of Proposition 3.1, we are going to obtain $\pi_t^u(X)$ as the value process Y of a BSDE with terminal condition $X \in L^\infty(\mathbb{P})$. Denoting by $\Pi_t(\cdot)$ and $\Pi_t^\perp(\cdot)$ the orthogonal projections on $\text{Im } \sigma_t^T$ and $\text{Ker } \sigma_t$, we have the following lemma; Its proof relies on suitable measurable selection arguments as well as modifications on $\mathbb{P} \otimes dt$ -nullsets; for details see [Ken15, Lems.2.14 and 2.22].

Lemma 5.10. For $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\tilde{\mu})$ there exists $\bar{\eta} = \bar{\eta}(Z, U)$ predictable and $\bar{\gamma} = \bar{\gamma}(Z, U)$ $\bar{\mathcal{P}}$ -measurable such that for $\mathbb{P} \otimes dt$ -almost all $(\omega, t) \in \Omega \times [0, T]$ holds

$$\bar{\eta}_t \Pi_t^\perp(Z_t) + \int_E U_t(e) \bar{\gamma}_t(e) \zeta_t(e) \lambda(de) = \max_{(\gamma, \eta) \in \bar{C}} \left(\eta_t \Pi_t^\perp(Z_t) + \int_E U_t(e) \gamma_t(e) \zeta_t(e) \lambda(de) \right), \quad (5.14)$$

where $\bar{C}_t = \left\{ (\gamma, \eta) \in L^2(\lambda_t) \times \mathbb{R}^d \mid \gamma \geq -1, \eta \in \text{Ker } \sigma_t, \|\gamma\|_{L^2(\lambda_t)}^2 + |\eta|_{\mathbb{R}^d}^2 \leq K_t^2 - |\varphi_t|_{\mathbb{R}^d}^2 \right\}$ is the closure of C_t in $L^2(\lambda_t) \times \mathbb{R}^d$ for any $t \leq T$.

To $(\bar{\gamma}, \bar{\eta}) \in \bar{C}$ of Lemma 5.10, we associate a probability measure $\bar{\mathbb{Q}} \ll \mathbb{P}$ defined via $d\bar{\mathbb{Q}} = \mathcal{E}((-\varphi + \bar{\eta}) \bullet B + \bar{\gamma} * \tilde{\mu}) d\mathbb{P}$, which may not be equivalent to \mathbb{P} as $\bar{\gamma}$ may be -1 on a non-negligible set. While $\bar{\mathbb{Q}}$ might not be in \mathcal{Q}^{ngd} it belongs to the $L^1(\mathbb{P})$ -closure of \mathcal{Q}^{ngd} in general, as shown in

Lemma 5.11. For $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\tilde{\mu})$, let $(\bar{\gamma}, \bar{\eta})$ be as in Lemma 5.10. Define the measures $\bar{\mathbb{Q}} \ll \mathbb{P}$ via $d\bar{\mathbb{Q}} = \mathcal{E}((-\varphi + \bar{\eta}) \bullet B + \bar{\gamma} * \tilde{\mu}) d\mathbb{P}$ and $\mathbb{Q}^n := (1/n)\hat{\mathbb{P}} + (1 - 1/n)\bar{\mathbb{Q}}$ for $n \in \mathbb{N}$. Then the densities $d\mathbb{Q}^n/d\mathbb{P}$ of the sequence $(\mathbb{Q}^n)_{n \in \mathbb{N}}$ in \mathcal{Q}^{ngd} converge to the one of $\bar{\mathbb{Q}}$ in $L^1(\mathbb{P})$ for $n \rightarrow \infty$. Consequently, $\pi_t^u(X) \geq \mathbb{E}_{\bar{\mathbb{Q}}}(X|\mathcal{F}_t)$ holds for all $t \leq T$.

Proof. Let $n \in \mathbb{N}$. Clearly $\mathbb{Q}^n \approx \mathbb{P}$. Moreover $d\mathbb{Q}^n/d\mathbb{P} = Z^n := (1/n)\hat{Z} + (1 - 1/n)\bar{Z}$ with $\hat{Z} := d\hat{\mathbb{P}}/d\mathbb{P} = \mathcal{E}(-\varphi \bullet B)$ and $\bar{Z} := d\bar{\mathbb{Q}}/d\mathbb{P}$. Itô formula then yields $Z^n = \mathcal{E}((-\varphi + \eta^n) \bullet B + \gamma^n * \tilde{\mu})$ for $\eta^n = \alpha \bar{\eta}$ being predictable and $\gamma^n = \alpha \bar{\gamma}$ is $\bar{\mathcal{P}}$ -measurable with $\alpha = (1 - 1/n)(\bar{Z}/Z^n) \in [0, 1]$ thanks to $\hat{Z} > 0$. Therefore $\eta^n \in \text{Ker } \sigma$ and $\gamma^n > -1$ due to $\bar{\gamma} \geq -1$. Hence $(\eta^n, \gamma^n) \in C$ and so $\mathbb{Q}^n = \mathbb{Q}^{\gamma^n, \eta^n}$ is in \mathcal{Q}^{ngd} . Convergence of \mathbb{Q}^n to $\bar{\mathbb{Q}}$ in $L^1(\mathbb{P})$ as $n \rightarrow \infty$ is straightforward by definition of \mathbb{Q}^n and this implies $\pi_t^u(X) \geq \mathbb{E}_{\bar{\mathbb{Q}}}(X|\mathcal{F}_t)$ for all $t \leq T$. \square

The dynamic good-deal bound $\pi^u(X)$ of $X \in L^\infty(\mathbb{P})$ is given by the solution to the JBSDE

$$-dY_t = \left((-\varphi_t + \bar{\eta}_t) Z_t + \int_E U_t(e) \bar{\gamma}_t(e) \zeta_t(e) \lambda(de) \right) dt - Z_t dB_t - \int_E U_t(e) \tilde{\mu}(dt, de), \quad t \in [0, T], \quad (5.15)$$

for terminal condition $Y_T = X$, with $\bar{\gamma} = \bar{\gamma}(Z, U)$, $\bar{\eta} = \bar{\eta}(Z, U)$ given by Lemma 5.10, according to

Theorem 5.12. For $X \in L^\infty(\mathbb{P})$, the JBSDE above with $(\bar{\gamma}, \bar{\eta})$ from (5.14) has a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$. Moreover there exists $\bar{\mathbb{Q}} \ll \mathbb{P}$ in the L^1 -closure of \mathcal{Q}^{ngd} (cf. Lemma 5.11), with density $d\bar{\mathbb{Q}}/d\mathbb{P} = \mathcal{E}((-\varphi + \bar{\eta}) \bullet B + \bar{\gamma} * \tilde{\mu})$ such that the good-deal bound satisfies

$$\pi_t^u(X) = \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}^{\text{ngd}}} \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = Y_t = \mathbb{E}_{\bar{\mathbb{Q}}}(X|\mathcal{F}_t) \quad \text{for } t \leq T. \quad (5.16)$$

Proof. Consider the family of BSDE generator functions defined for $(z, u) \in \mathbb{R}^d \times L^2(\zeta, d\lambda)$ by $f^{(\gamma, \eta)}(\cdot, z, u) := (-\varphi + \eta)z + \int_E u(e) \gamma(e) \zeta(e) \lambda(de)$ and $f^{(\gamma, \eta)}(\cdot, z, u) := 0$ elsewhere, for $(\gamma, \eta) \in \bar{C}$, where coefficients $(\gamma_t(\omega), -\varphi_t(\omega) + \eta_t(\omega))$ of $f^{(\gamma, \eta)}$ are bounded in $L^2(\lambda_t(\omega)) \times \mathbb{R}^d$ by $K_f := \|K\|_\infty \in (0, \infty)$ for all (γ, η) and (t, ω) . By Lemma 5.10, a classical generator function f for the JBSDE (5.15) can be defined such that $(\mathbb{P} \otimes dt\text{-a.e.}) f(\cdot, z, u) = \text{ess sup}_{(\gamma, \eta) \in \bar{C}} f^{(\gamma, \eta)}(\cdot, z, u)$ for all $(z, u) \in \mathbb{R}^d \times L^2(\zeta, d\lambda)$ and f is (a.e.) Lipschitz continuous in $(z, u) \in \mathbb{R}^d \times L^2(\lambda_t(\omega))$, with Lipschitz constant K_f . Indeed, such generator function f can be defined at first (up to a $\mathbb{P} \otimes dt$ -nullset) for countably many (z, u) with $z \in \mathbb{Q}^d$ and $u \in \{u^n, n \in \mathbb{N}\}$ dense subset of $L^2(\lambda)$ and, noting that $u \zeta_t(\omega)^{1/2}$ is in $L^2(\lambda)$ for u in $L^2(\lambda_t(\omega))$, by Lipschitz-continuous extension for all $(z, u) \in \mathbb{R}^d \times L^2(\lambda_t(\omega))$. By setting $f(t, z, u) := 0$ elsewhere (for $u \in L^0(\mathcal{B}(E), \lambda) \setminus L^2(\lambda_t(\omega))$), one can define f as Lipschitz continuous even for $(z, u) \in \mathbb{R}^d \times L^0(\mathcal{B}(E), \lambda)$.

By classical theory for Lipschitz-JBSDE, equation (5.15) thus has a unique solution (Y, Z, U) in $\mathcal{S}^2 \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$ which by boundedness of X satisfies $Y \in \mathcal{S}^\infty$ (cf. e.g. [Bec06, Prop.3.2-3.3]). Note that for all $(\gamma, \eta) \in \bar{C}$, clearly $\beta := -\varphi + \eta$ is bounded and $\int_E |\gamma_t(e)|^2 \zeta_t(e) \lambda(de)$ is bounded uniformly in $t \leq T$. Hence by Lemma 4.8, the BSDEs with generators $f^{\gamma, \eta}$ also have unique solutions $(Y^{\gamma, \eta}, Z^{\gamma, \eta}, U^{\gamma, \eta}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\tilde{\mu})$, which satisfy $Y_t^{\gamma, \eta} = \mathbb{E}_{\mathbb{Q}^{\gamma, \eta}}(X|\mathcal{F}_t)$, $\mathbb{Q}^{\gamma, \eta}$ -a.s., $t \leq T$. Since $f = f^{\bar{\gamma}, \bar{\eta}}$, we also have $Y_t = \mathbb{E}_{\bar{\mathbb{Q}}}(X|\mathcal{F}_t)$, $\bar{\mathbb{Q}}$ -a.s.. By Lemma 5.11 holds $\pi_t^u(X) \geq \mathbb{E}_{\bar{\mathbb{Q}}}(X|\mathcal{F}_t)$, $\bar{\mathbb{Q}}$ -a.s., for all $t \leq T$. To complete the proof, we show that $\pi_t^u(X) \leq Y_t$. For all $(\gamma, \eta) \in C$ (defining

$\mathbb{Q}^{\gamma,\eta} \in \mathcal{Q}^{\text{ngd}}$) we have that $f_t(Z_t, U_t) = f_t^{\bar{\gamma}, \bar{\eta}}(Z_t, U_t)$ dominates $f_t^{\gamma,\eta}(Z_t, U_t)$ for a.e. $t \leq T$. Noting that $f^{\gamma,\eta}$ are Lipschitz in (z, u) with (uniform) Lipschitz constant K_f and

$$f_t^{\gamma,\eta}(Z_t^{\gamma,\eta}, U_t) - f_t^{\gamma,\eta}(Z_t^{\gamma,\eta}, U_t^{\gamma,\eta}) = \int_E \gamma_t(e)(U_t(e) - U_t^{\gamma,\eta}(e))\zeta_t(e)\lambda(de), \quad t \leq T, \quad (5.17)$$

with $\mathcal{E}((-\varphi + \eta) \bullet B + \gamma * \tilde{\mu})$ being a martingale (see Example 3.3), one can apply comparison as in Proposition 3.1 to get $Y_t \geq Y_t^{\gamma,\eta}$, \mathbb{P} -a.s., $t \leq T$, $(\gamma, \eta) \in C$. Hence $Y_t \geq \text{ess sup}_{(\gamma,\eta)} Y_t^{\gamma,\eta} = \pi_t^u(X)$, $t \leq T$, for $(\gamma, -\varphi + \eta)$ ranging over all Girsanov kernels of measures $\mathbb{Q} \in \mathcal{Q}^{\text{ngd}}$. \square

References

- [BBP97] G. Barles, R. Buckdahn, and E. Pardoux. BSDE's and integral-partial differential equations. *Stochastics*, 60:57–83, 1997.
- [Bec06] D. Becherer. Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging. *Ann. Appl. Probab.*, 16:2027–2054, 2006.
- [Bec09] D. Becherer. From bounds on optimal growth towards a theory of good-deal hedging. In H. Albrecher, W. Runggaldier, and W. Schachermayer, editors, *Advanced Financial Modelling*, volume 8 of *Radon Series on Computational and Applied Mathematics*, pages 27–52. De Gruyter, Berlin, 2009.
- [Bec10] D. Becherer. Indifference prices and compensating variations. *Encyclopedia of Quantitative Finance*. Wiley, Chichester, 2010.
- [BEK09] P. Barrieu and N. El Karoui. Pricing, hedging and designing derivatives with risk measures. In R. Carmona, editor, *Indifference Pricing, Theory and Applications*, pages 77–146. Princeton Univ. Press, 2009.
- [BEK13] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.*, 41(3B):1831–1863, 2013.
- [BS05] D. Becherer and M. Schweizer. Classical solutions to reaction diffusion systems for hedging problems with interacting Itô and point processes. *Ann. Appl. Probab.*, 15:1111–1144, 2005.
- [BS06] T. Björk and I. Slinko. Towards a general theory of good-deal bounds. *Rev. Financ.*, 10:221–260, 2006.
- [CE10] S.N. Cohen and R. Elliott. Comparisons for backward stochastic differential equations on Markov chains and related no-arbitrage conditions. *Ann. Appl. Probab.*, 20:267–311, 2010.
- [CF14] F. Confortola and M. Fuhrman. Backward stochastic differential equations associated to jump Markov processes and applications. *Stochastic Process. Appl.*, 124:289–316, 2014.
- [CFJ16] F. Confortola, M. Fuhrman, and J. Jacod. Backward stochastic differential equations driven by a marked point process: an elementary approach with an application to optimal control. *Ann. Appl. Probab.*, 26(3):1743–1773, 2016.
- [CGM⁺02] P. Carr, H. Geman, D. Madan, , and M. Yor. The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75(2):305 – 332, 2002.
- [CM08] S. Crépey and A. Matoussi. Reflected and doubly reflected BSDE with jumps: A priori estimates and comparison. *Ann. Appl. Probab.*, 18:2045–2069, 2008.
- [CSR00] J. Cochrane and J. Saá Requejo. Beyond arbitrage: Good deal asset price bounds in incomplete markets. *J. Polit. Econ.*, 108:79–119, 2000.

- [Del06] F. Delbaen. The structure of m -stable sets and in particular of the set of risk neutral measures. In *Séminaire Probabilités 39*, Lecture Notes in Mathematics 1874, pages 215–258. Springer, Berlin, 2006.
- [DI10] L. Delong and P. Imkeller. On Malliavin’s differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. *Stochastic Process. Appl.*, 120(1):1748–1775, 2010.
- [DM82] C. Dellacherie and P. Meyer. *Probabilities and Potential B, Theory of Martingales*. Mathematics Studies. North Holland, Amsterdam, New York, Oxford, 1982.
- [DP15] L. Delong and A. Pelsser. Instantaneous mean-variance hedging and Sharpe ratio pricing in a regime-switching financial model. *Stochastic Models*, 31:67–97, 2015.
- [DTE15a] P. Di Tella and H.-J. Engelbert. On the predictable representation property of martingales associated with Lévy processes. *Stochastics*, 87(1):1–15, 2015.
- [DTE15b] P. Di Tella and H.-J. Engelbert. The predictable representation property of compensated-covariation stable families of martingales. *Theory Probab. Appl.*, 60:99–130, 2015.
- [EMN16] N. El Karoui, A. Matoussi, and A. Ngoupeyou. Quadratic exponential semimartingales and application to BSDEs with jumps. *arXiv:1603.06191 preprint*, 2016.
- [GL16] C. Geiss and C. Labart. Simulation of BSDEs with jumps by Wiener Chaos expansion. *Stochastic Process. Appl.*, 126(7):2123–2162, 2016.
- [GS16] C. Geiss and A. Steinicke. L_2 -variation of Lévy driven BSDEs with non-smooth terminal conditions. *Bernoulli*, 22(2):995–1025, 2016.
- [HIM05] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, 15:1691–1712, 2005.
- [HWY92] S. He, J. Wang, and J. Yan. *Semimartingale Theory and Stochastic Calculus*. Science Press, CRC Press, New York, 1992.
- [JS03] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, 2003.
- [Kaz79] N. Kazamaki. A sufficient condition for the uniform integrability of exponential martingales. *Mathematical Reports Toyama University*, 2:1 – 11, 1979.
- [Kaz94] N. Kazamaki. *Continuous Exponential Martingales and BMO*. Lecture Notes in Mathematics 1579. Springer, Berlin, 1994.
- [Ken15] K. Kentia. *Robust aspects of hedging and valuation in incomplete markets and related backward SDE theory*. PhD thesis, Humboldt-Universität zu Berlin, 2015. [urn:nbn:de:kobv:11-100237580](https://nbn-resolving.org/urn:nbn:de:kobv:11-100237580).
- [KMPZ10] I. Kharroubi, J. Ma, H. Pham, and J. Zhang. Backward SDEs with constrained jumps and quasi-variational inequalities. *Ann. Appl. Probab.*, 38:794–840, 2010.
- [Kob00] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Appl. Probab.*, 28:558–602, 2000.
- [KTPZ15] M.N. Kazi-Tani, D. Possamaï, and C. Zhou. Quadratic BSDEs with jumps: a fixed-point approach. *Electronic Journal of Probability*, 20(66):1–28, 2015.
- [KTPZ16] N. Kazi-Tani, D. Possamaï, and C. Zhou. Quadratic BSDEs with jumps: Related nonlinear expectations. *Stoch. Dyn.*, 16(4):1650012, 32, 2016.
- [LM78] D. Lepingle and J. Mémin. Sur l’intégrabilité uniforme des martingales exponentielles. *Z. Wahrscheinlichkeitstheor. verw. Geb.*, 42:175–203, 1978.

- [LS14] R. J. A. Laeven and M. A. Stadjé. Robust portfolio choice and indifference valuation. *Mathematics of Operations Research*, 39:1109–1141, 2014.
- [MC14] M. Mania and B. Chikvinidze. New proofs of some results on BMO martingales using BSDEs. *Journal of Theoretical Probability*, 27:1213–1228, 2014.
- [Mor09] M. Morlais. Utility maximization in a jump market model. *Stochastics*, 81:1–27, 2009.
- [Mor10] M. Morlais. A new existence result for quadratic bsdes with jumps with application to the utility maximization problem. *Stochastic Process. Appl.*, 120:1966–1995, 2010.
- [NS00] D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stochastic Process. Appl.*, 90:109–122, 2000.
- [NS01] D. Nualart and W. Schoutens. Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance. *Bernoulli*, 7(5):761–776, 2001.
- [Nut10] M. Nutz. The opportunity process for optimal consumption and investment with power utility. *Math. Financ. Econ.*, 3(3-4):139–159, 2010.
- [Nut12] M. Nutz. The Bellman equation for power utility maximization with semimartingales. *Ann. Appl. Probab.*, 22:363–406, 2012.
- [Par97] E. Pardoux. Generalized discontinuous backward stochastic differential equations. In El Karoui N. and L. Mazliak, editors, *Backward Stochastic Differential Equations*, Pitman Research Notes in Mathematical Sciences 364, pages 209–219. Longman, 1997.
- [PP90] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *System Control Lett.*, 14:55–61, 1990.
- [PS08] P. Protter and K. Shimbo. No arbitrage and general semimartingales. *Markov Processes and related Topics: A Festschrift for Thomas G. Kurtz*, 4:267–283, 2008.
- [QS13] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Process. Appl.*, 123(8):3328–3357, 2013.
- [Roc76] R. T. Rockafellar. Integral functionals, normal integrands and measurable selections. In L. Waelbroeck, editor, *Nonlinear Operators and Calculus of Variations*, Lecture Notes in Mathematics 543, pages 157–207. Springer, Berlin, 1976.
- [Roy06] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Process. Appl.*, 116:1358–1376, 2006.
- [Sek06] J. Sekine. On exponential hedging and related quadratic backward stochastic differential equations. *Appl. Math. Optim.*, 54:131–158, 2006.
- [Tev08] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Process. Appl.*, 118(3):503–515, 2008.
- [TL94] S. Tang and X. Li. Necessary conditions for optimal control for stochastic systems with random jumps. *SIAM J. Control Optim.*, 32:1447–1475, 1994.